Entry-Proofness and Discriminatory Pricing under Adverse Selection*

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Abstract

This paper studies competitive allocations under adverse selection. We first provide a general necessary and sufficient condition for entry on an inactive market to be unprofitable. We then use this result to characterize, in an active market, a unique budget-balanced allocation implemented by a market tariff making additional trades with an entrant unprofitable. Motivated by the recursive structure of this allocation, we finally show that it emerges as the essentially unique equilibrium outcome of a discriminatory ascending auction. These results yield sharp predictions for competitive nonexclusive markets.

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1 Introduction

It has long been recognized that markets subject to adverse selection can unravel to a no-trade equilibrium. As shown by Akerlof (1970), this can occur even if trade would always be mutually beneficial if the quality of the goods for sale were commonly known. This failure of the price mechanism has been recently invoked to explain phenomena such as insurance rejections (Hendren (2013)) and to justify public intervention in the presence of liquidity or credit freezes (Philippon and Skreta (2012), Tirole (2012)). In this paper, we build on this insight to provide a new approach to the characterization of competitive allocations under adverse selection, based on the standard premise that a perfectly competitive market should be immune to entry.

To this end, we consider a general adverse-selection economy in which little structure is imposed on buyers' preferences. This setting encompasses insurance economies as well as standard trade environments, with or without wealth effects. The main restrictions are a single-crossing condition and a monotonicity condition on costs implying *weak adverse selection*, in the sense that buyer types who are more willing to make larger purchases are *on average* more costly to serve. The corresponding expected costs turn out to play a key role in the description of market outcomes, as they do in Akerlof (1970).

In this context, entry-proofness provides a tractable and detail-free alternative to the strategic approaches adopted in the literature. We apply this requirement to prove two theorems that respectively pertain to inactive and active markets, depending on whether or not trade opportunities are available on the market. At the core of our approach is a unified treatment of these two cases.

Theorem 1 states a necessary and sufficient condition for entry to be unprofitable in an inactive market, which generalizes the market-unraveling condition first formulated by Akerlof (1970) and recently extended by Hendren (2013) to Rothschild and Stiglitz (1976) insurance economies. The intuition is as follows. Under weak adverse selection, the cost of selling a unit of the good depends on the buyer types who purchase it; on average, the cost is the upper-tail conditional expectation of unit costs, starting from the first type who purchases this unit. Our entry-proofness (EP) condition then simply states that the willingness to pay of each type at the no-trade point should not exceed this cost. The necessity of Condition EP for entry-proofness is straightforward and only requires the use of single-contract offers. By contrast, its sufficiency must account for an entrant's ability to offer a menu of contracts. To complete the proof of Theorem 1, we identify a weak assumption on buyers' preferences under which entry with a menu of contracts is unprofitable as soon as entry with a single contract is unprofitable.

We next consider active markets, on which trade opportunities are available. Rothschild and Stiglitz (1976) have characterized the set of *exclusive* contracts that prevent an entrant from making a profit. However, in many financial and insurance markets, a seller cannot monitor the trades a given buyer makes with his competitors. A natural question is thus to characterize the outcomes of *nonexclusive* markets that meet the requirement that entry be unprofitable. This leads us to define a set of contracts, or a *market tariff*, as *entry-proof* if it prevents an entrant from profitably offering a menu of contracts that complements the tariff in the sense that buyers are free to combine any contract offered by the entrant with any trade along the tariff.

Nonexclusivity makes screening difficult as sellers do not observe the aggregate quantity purchased by any buyer. In particular, each seller anticipates that a buyer may purchase a large quantity by splitting it between several sellers. Under weak adverse selection, this is a concern for him, because these additional trading opportunities are especially attractive for buyers who happen to be on average more costly to serve. One way to hedge against this risk is to offer a convex tariff that prices successive marginal quantities at an increasing rate. An illustration is a discriminatory limit-order book, such as the NASDAQ, in which market makers place limit orders that are executed in order of price priority. This motivates us to focus on convex market tariffs in the bulk of our analysis; we argue in Section 6 that this assumption can be significantly relaxed without affecting our results.

The convexity assumption is analytically convenient because it allows us to characterize entry-proof markets tariffs by building on the results derived for inactive markets. The basic idea is to factor all available trade opportunities into the buyers' preferences. Indeed, from an entrant's viewpoint, everything happens as if he were facing an inactive market in which the buyers' preferences for any trade he may make available are represented by indirect utility functions incorporating their optimal trades along the market tariff. The key point is that, when this tariff is convex, these indirect utility functions inherit single-crossing from the primitive utility functions. This, in turn, implies that the entrant faces weak adverse selection, exactly as in an inactive market. More technically, convexity of the market tariff ensures that the buyers' indirect utility functions satisfy all the regularity properties needed to apply Theorem 1.

Theorem 2 singles out a unique budget-feasible allocation implemented by an entry-proof convex market tariff, and an essentially unique such tariff; existence obtains under very general conditions, unlike in the exclusive-competition case. The market tariff is typically nonlinear, contradicting a common presumption that nonexclusivity should lead to linear pricing. Thus pricing is discriminatory, in the sense that the successive quantity *layers* offered along this tariff are priced at different rates. Specifically, each layer is priced at the expected cost of serving the types who optimally choose to purchase it, so that the corresponding profit is zero; under weak adverse selection, this cost is equal to the upper-tail conditional expectation of unit costs, starting from the marginal type. When the buyers' preferences are linear, subject to a capacity constraint, these properties lead to Akerlof (1970) pricing and to the competitive-equilibrium allocation that maximizes the gains from trade. When the buyers' preferences are strictly convex, they lead to a marginal version of Akerlof (1970) pricing and to an allocation generalizing those highlighted, in specific contexts, by Jaynes (1978), Hellwig (1988), and Glosten (1994). We will accordingly refer to the *JHG allocation* and to the *JHG tariff*.

A noticeable feature of the JHG allocation is its recursive structure. On the first layer, the price is the expected cost of serving all types, and the quantity purchased by each of them is exactly the demand of the first type at this price. Indeed, supplying less would inefficiently ration demand, while supplying more would entail losses on the excess quantity. On the second layer, the first type is no longer active, and the same reasoning applies: the price is the expected cost of serving all types except the first, and the quantity purchased by each of them is the residual demand of the second type at this price—and so on. Overall, on each layer, what sellers are ready to supply exactly matches the residual demand of the marginal type. Thus supply equals marginal residual demand on each layer, at a price equal to expected cost. In that sense, the JHG allocation is competitive.

The existence, uniqueness, and competitive features of the JHG allocation are arguably strong arguments in favor of using entry-proofness as a conceptual tool for predicting the outcomes of nonexclusive markets under adverse selection. However, this approach remains silent on how to implement this allocation in a decentralized way, because it does not explain how the JHG tariff comes into existence. It is thus natural to ask whether the JHG allocation and the JHG tariff can be derived as the aggregate equilibrium outcome of a game in which strategic sellers compete to serve the buyers' demand.

In this respect, the recursive structure of the JHG allocation suggests a setting in which competition takes place sequentially, layer by layer. To validate this intuition, we model the strategic interactions between sellers as a discriminatory ascending auction. Prices are quoted sequentially, in increasing order, and according to a discrete price grid with a minimum tick size. Each time a new price is quoted, each seller publicly announces the maximum quantity he stands ready to trade with each buyer at this price; that is, he places a limit order at this price. Once this process is completed, each buyer selects from the limit-order book sequentially built in this way the orders she wishes to execute, according to her type. As it is optimal for her to start with the best price offers, she effectively faces a convex market tariff. Trading is nonexclusive in the sense that each buyer can simultaneously trade with several sellers.

These simple trading rules define a standard extensive-form game with almost-perfect information. Our main results are encapsulated in two theorems. Theorem 3 exhibits a simple equilibrium in which, at each price and in each subgame, sellers equally share the buyers' profitable residual demand. By construction, the resulting aggregate equilibrium allocation converges to the JHG allocation when the tick size goes to zero. Theorem 4 then reinforces this result, by showing that, modulo a natural refinement, any sequence of aggregate equilibrium allocations converges to the JHG allocation when the tick size goes to zero. Thus the JHG allocation emerges as the essentially unique outcome of competition when each seller can quickly react to his competitors' offers. These positive results, which stand in stark contrast with the pervasive nonexistence results that plague the literature on competitive screening under adverse selection, invite us to reconsider the role of sequential trading for financial and insurance markets.

Contributions to the Literature

Theorem 1 generalizes results obtained by Akerlof (1970), Glosten (1994), and Mailath and Nöldeke (2008) in the quasilinear case, and by Hendren (2013) in the case of a Rothschild and Stiglitz (1976) economy. Our contribution is to state a general necessary and sufficient condition for an inactive market to be entry-proof, to point out a technical condition on preferences that has been so far overlooked, and to provide a comprehensive yet elementary proof that may be useful for pedagogical purposes. We also argue that, under strict single-crossing, Condition EP is necessary and sufficient for entry-proofness even when selection is not adverse, so that the ordering of costs with respect to types is arbitrary. This significant extension of the scope of Hendren's (2013) result may be relevant for the empirical study of insurance markets where consumers differ both in riskiness and in risk aversion.

The unique allocation that survives entry in a nonexclusive market in which supply is described by a convex tariff corresponds to the allocations characterized by Akerlof (1970) in the case of an indivisible good, and by Jaynes (1978), Hellwig (1988), and Glosten (1994) in the case of a divisible good. Beyond extending these results to general preferences, our contribution is to apply our results on inactive markets to active markets, exploiting the idea that a nonexclusive tariff is entry-proof if and only if no additional trades are both incentive-feasible and profitable. This original approach allows for a unified treatment of linear and strictly convex preferences.

Entry-proofness in exclusive markets has been well understood since Rothschild and Stiglitz (1976). The unique candidate is the Riley (1979) allocation, characterized by the absence of cross-subsidies between types and downward-binding local incentive-compatibility constraints. However, this allocation generally fails to be entry-proof when there are many types (Riley (1985)). We argue in Section 6 that the main difference with the nonexclusive markets studied in this paper is that the buyers' indirect utility functions induced by an exclusive tariff do not satisfy single-crossing, so that an entrant can engage in cream-skimming without worrying about adverse selection. By contrast, single-crossing is satisfied under nonexclusivity as long as the market tariff is convex; as a result, cream-skimming is impossible and adverse selection is unavoidable. This explains why we are able to obtain a general existence result.

Despite the renewed interest for competitive nonexclusive markets under adverse selection, the literature has not yet delivered a sharp prediction for the corresponding market outcomes. The early works of Bisin and Gottardi (1999, 2003) establish the existence of competitive equilibria when bid-ask spreads or entry fees are allowed for, but do not characterize these equilibria. Subsequent oligopolistic approaches have made more concrete steps towards the characterization of equilibrium trades by considering competitive-screening games in which sellers simultaneously offer menus of contracts, or tariffs, from which a buyer is free to choose according to her private information.

In this spirit, Biais, Martimort, and Rochet (2000) construct an equilibrium in convex tariffs in a setting where the buyer has strictly convex preferences and the distribution of types is continuous. The aggregate equilibrium tariff is not entry-proof, but it converges to the JHG tariff when the number of sellers grows large. This sounds promising, but Attar, Mariotti, and Salanié (2014, 2019) argue that slightly perturbing this model by discretizing the distribution of types leads to a completely different picture: the JHG allocation becomes the unique candidate-equilibrium allocation, but it can be supported in equilibrium only in the extreme case where it features a single layer.[1] These discontinuity and existence problems

[1]When the buyer's preferences are linear subject to a capacity constraint, Attar, Mariotti, and Salanié (2011) show that, for any distribution of types, the Akerlof (1970) competitive-equilibrium allocation that maximizes the gains from trade is the unique aggregate equilibrium allocation, and that it can be sustained in equilibrium by each seller posting the JHG tariff, which in this case consists of a single layer.

make the equilibrium predictions of competitive-screening games somewhat fragile, as they ultimately hinge on fine modeling details.[2] By contrast, focusing on entry-proof market tariffs has enabled us to derive a sharp and robust prediction for nonexclusive competitive markets. Existence and uniqueness hold under fairly general assumptions, and the JHG allocation may be seen as a compelling extension of Akerlof (1970) to the case of a divisible good and general preferences.

The recursive structure of the JHG allocation has motivated us to design an ascending discriminatory auction in which the market tariff is built sequentially. This contrasts with the competitive-screening games studied by Biais, Martimort, and Rochet (2000), Back and Baruch (2013), and Attar, Mariotti and Salanié (2019), which can be interpreted as discriminatory auctions in which sellers simultaneously bid at all prices. The advantage of a sequential auction lies in its transparency, a point that has been emphasized in other contexts by Milgrom (2000) and Ausubel (2004): each seller can directly react at each stage of the auctioning phase to the past supply decisions of his competitors. This allows for a richer set of punishments than in competitive-screening games, where deviations can be punished only through the buyers' decisions. Our contribution is to provide a fully strategic foundation for the JHG allocation, a result that has so far eluded the literature.[3]

An alternative derivation of the JHG allocation is provided by Beaudry and Poitevin (1995), who study a sequential game in which a risk-averse entrepreneur whose project can be of low or high riskiness can repeatedly solicit financing from successive cohorts of uninformed lenders, thereby signaling the type of her project. In comparison, a realistic feature of our setting is that the set of sellers is fixed throughout the auctioning phase, so that each seller must anticipate the future consequences of his supply decisions at any price. Moreover, while signaling is an integral part of Beaudry and Poitevin (1995) and requires an appropriate selection of lenders' beliefs off the equilibrium path, it plays no role in our analysis as buyers choose their optimal quantities only once all offers have been made.

The paper is organized as follows. Section 2 describes the model. Section 3 analyzes inactive markets. Section 4 extends the analysis to active markets. Section 5 studies the

[2]Back and Baruch (2013) and Biais, Martimort, and Rochet (2013) further argue that the equilibrium constructed by Biais, Martimort, and Rochet (2000) only exists under rather stringent joint restrictions on the cost function and the distribution of types.

[3]To be fair, Attar, Mariotti, and Salanié (2019) show that, as the number K of sellers grows large, a standard competitive-screening game admits an ε-equilibrium, with ε of the order of $1/K^2$, that supports the JHG allocation. The results in this paper are significantly stronger in that they rely neither on a notion of approximate equilibrium nor on the consideration of a fictitious competitive limit, and deliver a new perspective on market design by showing that a sequential auction implements the JHG allocation as an essentially unique equilibrium outcome.

discriminatory ascending auction. Section 6 discusses our results. Section 7 concludes. The main appendix provides the proofs of Theorems 1–4. The online appendices A through D collect supplementary material.

2 The Economy

Consider a buyer (she) endowed with private information, and whose type $i = 1, \ldots, I$ can take a finite number of values with strictly positive probabilities m_i; alternatively, we can view the economy as populated by a continuum of buyers, a proportion m_i of which is of type i. Type i's preferences are represented by a utility function $u_i(q, t)$ that is continuous and weakly quasiconcave in (q, t) and strictly decreasing in t, with the interpretation that q is the nonnegative quantity of a divisible good she purchases and t is the payment she makes in return. Types are ordered according to the weak single-crossing condition (Milgrom and Shannon (1994)), which states that higher types are at least as willing to increase their purchases than lower types are:

$$\text{For all } i < j,\ q < q',\ t, \text{ and } t',\ u_i(q, t) \leq (<)\, u_i(q', t') \text{ implies } u_j(q, t) \leq (<)\, u_j(q', t').$$

For future reference, we also state the slightly stronger, strict single-crossing condition:

$$\text{For all } i < j,\ q < q',\ t, \text{ and } t',\ u_i(q, t) \leq u_i(q', t') \text{ implies } u_j(q, t) < u_j(q', t').$$

To define marginal rates of substitution without assuming differentiability, let $\tau_i(q, t)$ be the supremum of the set of prices p such that

$$u_i(q, t) < \max\{u_i(q + q', t + pq') : q' \geq 0\}.$$

Thus $\tau_i(q, t)$ is the slope of type i's indifference curve at the right of (q, t). Quasiconcavity ensures that $\tau_i(q, t)$ is finite, except possibly when $q = 0$, and that it is nonincreasing along an indifference curve of type i. We additionally make the intuitive assumption that, in the absence of transfers, a positive endowment of q reduces this marginal rate of substitution.

Assumption 1 *For all i and $q > 0$, $\tau_i(q, 0) \leq \tau_i(0, 0)$.*

Our assumptions on the buyer's preferences hold in a Rothschild and Stiglitz (1976) insurance economy, which is the case studied by Hendren (2013); then i indexes the buyer's riskiness, q is the amount of coverage she purchases, and t is the premium she pays in return. As we illustrate in Appendix C, they also hold under many alternative specifications, allowing for multiple loss levels or various forms of nonexpected utility. Finally, they encompass a broad variety of other applications, such as financial and labor markets. It should be noted that we do not require strict single-crossing nor strict convexity of preferences. This choice is not motivated by an idle desire for generality, but is meant to pave the way for the analysis of active markets provided in Section 4.

The supply side of the economy is represented by a linear technology, with unit cost $c_i > 0$ when the buyer's type is i. For each i, we denote by $\overline{c}_i$ the upper-tail conditional expectation of unit costs,

$$\overline{c}_i \equiv \mathbf{E}[c_j \,|\, j \geq i] = \frac{\sum_{j \geq i} m_j c_j}{\sum_{j \geq i} m_j}.$$

Adverse selection occurs if the unit cost c_i is nondecreasing in i. Here, and unless indicated otherwise, we make the slightly weaker assumption that $\overline{c}_i$ is nondecreasing in i. This weak adverse-selection condition is exactly equivalent to

$$\text{For all } j \leq i,\ c_j \leq \overline{c}_i. \tag{1}$$

Each seller (he) is risk-neutral and thus maximize his expected profit.

In the continuum-of-buyers interpretation of the model, contracting is bilateral and nonanonymous. Thus each seller fully monitors the trades each buyer makes with him, allowing him to charge a different price for different marginal units. We make the standard assumption that buyers of the same type facing the same choices behave in the same way.

3 Entry-Proofness in Inactive Markets

In this section, we describe the circumstances under which private information impedes trade altogether. We shall adopt the following terminology. A *contract* is a pair (q, t) for some nonegative q, and the *null contract* is the pair $(0, 0)$. A market is *inactive* if, for whichever reason, only the null contract is available. An inactive market is *entry-proof* if and only if *for any menu of contracts offered by an entrant, the buyer has a best response such that the entrant earns at most zero expected profit.* Later in this section, we will strengthen this definition in relation to the stronger notion of market breakdown. Our first task is to characterize markets that are both inactive and entry-proof.

Let us first analyze the simple case where the entrant offers a single contract, designed so as to attract some type i. To do so, the entrant can choose some unit price p slightly below $\tau_i(0, 0)$. Then, by definition of $\tau_i(0, 0)$, there exists a quantity q that strictly attracts type i at this price, that is, $u_i(q, pq) > u_i(0, 0)$. As types are ordered according to the weak single-crossing condition, we also have $u_j(q, pq) > u_j(0, 0)$ for all $j > i$. Thus any type $j \geq i$ is strictly attracted by the offer (q, pq), and the entrant bears an expected unit cost $\overline{c}_i$ when trading with these types. Finally, some other types $j < i$ may also be attracted, but (1) ensures that this can only reduce the entrant's expected unit cost.[4] This simple reasoning shows that the following condition is necessary for entry to be unprofitable.

Condition EP *For each* i, $\tau_i(0, 0) \leq \overline{c}_i$.

The following theorem, a formal proof of which is provided in the main appendix, states that this necessary condition is also sufficient, even when menus of contracts are allowed.

Theorem 1 *An inactive market is entry-proof if and only if Condition EP is satisfied.*

The key to the proof lies in the following remark. Suppose the entrant offers an arbitrary menu of contracts. Under weak single-crossing, a standard monotone-comparative-statics argument implies that the buyer has a best response with nondecreasing quantities; that is, the entrant ends up trading (q_i, t_i) with every type i, with $q_i \leq q_j$ for all $i < j$. Then his expected profit is

$$\sum_i m_i(t_i - c_i q_i),$$

which, using a summation by parts in the spirit of Wilson (1993), we can rewrite as

$$\sum_i \left(\sum_{j \geq i} m_j \right) [t_i - t_{i-1} - \overline{c}_i(q_i - q_{i-1})], \tag{2}$$

where $(q_0, t_0) \equiv (0, 0)$. Now, because type i is willing to trade $(q_i - q_{i-1}, t_i - t_{i-1})$ in addition to (q_{i-1}, t_{i-1}), each bracketed term in (2) cannot exceed

$$[\tau_i(q_{i-1}, t_{i-1}) - \overline{c}_i](q_i - q_{i-1}),$$

and thus, as $q_i \geq q_{i-1}$, will be nonpositive if the marginal rate of substitution is lower than the upper-tail conditional expectation of unit costs,

$$\tau_i(q_{i-1}, t_{i-1}) \leq \overline{c}_i.$$

To show that this holds, recall that, by construction, type $i - 1$ prefers her optimal choice (q_{i-1}, t_{i-1}) to the no-trade contract $(0, 0)$. Under weak single-crossing, the same property is

[4] Alternatively, if we were to assume strict single-crossing, then we could design (q, pq) so that types below i are not attracted. Then assumption (1) on costs would not be needed anymore for Theorem 1 to hold.

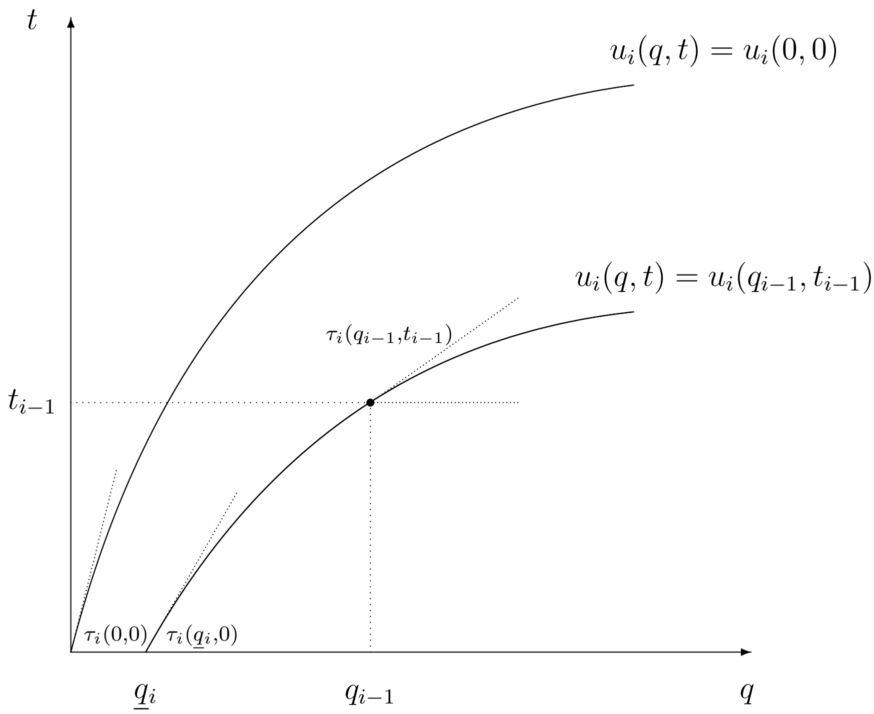

Figure 1: A graphical illustration of (3).

satisfied by type i, and thus (q_{i-1}, t_{i-1}) lies in the nonnegative orthant, below the indifference curve of type i that goes through the origin. (That we can focus on menus with nonnegative transfers is established in the main appendix.) As illustrated in Figure 1, we can apply in turn the concavity of the indifference curve of type i, then Assumption 1, and finally Condition EP to obtain the desired inequality:

$$\tau_i(q_{i-1}, t_{i-1}) \leq \tau_i(\underline{q}_i, 0) \leq \tau_i(0,0) \leq \overline{c}_i. \tag{3}$$

This concludes the proof of Theorem 1.

A noticeable feature of this proof is that it does not consider each contract (q_i, t_i) in isolation. Instead, the key role is played by *layers* of the form $(q_i - q_{i-1}, t_i - t_{i-1})$. Under weak single-crossing, the i^{th} layer can be thought of as traded by all types $j \geq i$, and thus has expected unit cost $\overline{c}_i$. Condition EP then states that, at this price, type i is not strictly willing to trade, so that each layer must yield a nonpositive expected profit. By contrast, some of the *contracts* proposed in a menu may yield positive profits. For example, although the condition $t_1 \leq \overline{c}_1 q_1$ ensures that the expected profit on the first layer (q_1, t_1) is nonpositive, it may well be that $t_1 > c_1 q_1$. Hence Condition EP does not rule out gains from trade, in the usual first-best sense of the term.

The assumptions of Theorem 1 can be weakened in three directions. First, the finiteness of the type distribution is not crucial: we show in Appendix B that the result holds for an arbitrary type distribution with bounded support over the real line. Second, the convexity of preferences can be relaxed if we reinforce Assumption 1 into $\tau_i(q,t) \leq \tau_i(0,0)$ for all (q,t) such that $t \geq 0$ and $u_i(q,t) \geq u_i(0,0)$. Third, the monotonicity assumption (1) on costs is not needed if we assume strict single-crossing.[5] Thus, whereas our result relies on private information, it does not require adverse selection, even in the weak form (1). This makes it applicable when selection is advantageous (Hemenway (1990)) or in any intermediate case, a notable feature in light of the literature that questions the empirical relevance of adverse selection (Finkelstein and McGarry (2006), Fang, Keane, and Silverman (2008)).

By contrast, the weak single-crossing condition and the seemingly innocuous Assumption 1 are tight. The key role of the weak single-crossing condition in the proof of Theorem 1 is to ensure that the quantity profile chosen by the buyer in the entrant's menu is nondecreasing in her type. As for Assumption 1, Example 1 in Appendix D shows that, in its absence, entry with a menu of contracts can be profitable even though Condition EP is satisfied—the intuition being that type i's marginal rate of substitution can then take values higher than $\tau_i(0,0)$ in the relevant area illustrated in Figure 1.

Condition EP ensures that there exists a best response for the buyer such that entry on an inactive market is unprofitable. However, the literature often focuses on characterizing *market breakdown*, defined as a situation in which *any menu of non-null contracts yields a strictly negative expected profit, even if the buyer's best response is most favorable to the entrant*. Condition EP clearly remains necessary for this stronger concept. We now argue that, under slightly stronger conditions on preferences, it remains also sufficient. The proof of the following result is provided in the main appendix, and Appendix D provides two examples showing that the additional conditions are tight.

Corollary 1 *Suppose that the buyer's preferences are strictly convex and that types are ordered according to the strict single-crossing condition. Then there is market breakdown if and only if Condition EP is satisfied.*

Mailath and Nöldeke (2008) obtain a related result for an economy in which the buyer has quadratic quasilinear preferences, as in Glosten (1989), Biais, Martimort, and Rochet (2000, 2013), and Back and Baruch (2013). Yet they focus on competitive pricing, defined as a situation in which each quantity traded yields zero profit, so that there are no cross-subsidies between contracts. As noticed above, this is an important restriction: Example 3 in Appendix D shows that the best pricing strategy for the entrant may not be competitive in this sense, because what matters for entry-proofness is not so much the profit earned on each quantity q_i than the expected profit earned on each quantity layer $q_i - q_{i-1}$.

[5] See Footnotes 4 and 15 for the only change needed in the proof.

Hendren (2013) studies a Rothschild and Stiglitz (1976) insurance economy, and his Theorem 1 is the analogue of Corollary 1 in this particular setting. As emphasized by the author, an implication of Condition EP is that the highest-risk type I must not be willing to purchase coverage at the actuarially fair rate c_I. Given that her preferences have an expected-utility representation, this is possible only if type I incurs a loss with certainty. In that case, type I's preferences are no longer strictly convex, and the above result becomes that all types except perhaps type I must be excluded from trade, as in Akerlof's (1970) classic example of market breakdown.

4 Entry-Proofness in Active Markets

We now turn to *active* markets, in which nonnull contracts are available. We follow Rothschild and Stiglitz (1976) and characterize the set of contracts that deters subsequent entry. In contrast with them, however, we focus on situations in which the buyer cannot be prevented from trading with more than one seller—that is, trade is *nonexclusive*. In nonexclusive markets, each seller aims at limiting the risk of attracting high-cost types buying large quantities, and to do so can place *limit orders*—that is, offers to sell at a given price up to a maximum quantity. Perhaps for this reason, limit orders are one of the main instruments used on financial markets, and especially so when the market is organized as a limit-order book (Glosten (1994)). The key property that we exploit below is that, if sellers on the market place collections of limit orders, the buyer faces a convex market tariff T, obtained by convoluting these orders. We use Condition EP to show that requiring that such a tariff be entry-proof singles out a unique budget-feasible allocation, the construction of which crucially hinges on upper-tail conditional expectations of unit costs.

We throughout assume that the domain of the convex tariff T is a compact interval containing 0, with $T(0) \equiv 0$. Every type i selects q_i so as to maximize $u_i(q, T(q))$. We then say that the allocation $(q_i, T(q_i))_{i=1}^I$ is *implemented* by the tariff T, and that it is *budget-feasible* if

$$\sum_i m_i[T(q_i) - c_i q_i] \geq 0. \tag{4}$$

We also assume that types are ordered according to the strict single-crossing condition. As a result, in any allocation implemented by the tariff T, the optimal quantities q_i are nondecreasing in i.

Now, suppose an entrant can propose additional trades to the buyer, in the form of a menu of contracts. We say that the tariff T is *entry-proof* if *for any menu of contracts offered* *by an entrant, the buyer has a best response such that the entrant earns at most zero expected profit, taking into account that the buyer is free to combine the entrant's contracts with those made available by the tariff* T. The last clause of this definition is crucial, and captures the nonexclusive nature of trade. Our goal is to characterize the set of budget-feasible allocations that are implementable by entry-proof convex market tariffs.

Let us first observe that, from the entrant's viewpoint, everything happens as if he were facing modified types with indirect utility functions

$$u_i^T(q',t') \equiv \max\{u_i(q+q',T(q)+t') : q\}, \tag{5}$$

reflecting that the buyer is free to combine any contract (q',t') offered by the entrant with a trade along the tariff T. In particular, $u_i^T(0,0)$ represents type i's utility when she only trades on the market and not with the entrant, and thus defines the relevant individual-rationality constraint for type i from the entrant's viewpoint.[6]

Because the tariff T is continuous over a compact domain, the maximum in (5) is attained and $u_i^T(q',t')$ is continuous in (q',t').[7] Moreover, because the tariff T is convex and the primitive utility functions $u_i(q,t)$ are weakly quasiconcave in (q,t) and strictly decreasing in t, the indirect utility functions $u_i^T(q',t')$ are weakly quasiconcave in (q',t') and strictly decreasing in t'. As a result, we can define the marginal rates of substitution $\tau_i^T(q',t')$ associated to them exactly as we did in Section 2 for the primitive utility functions. Finally, because the primitive types are ordered according to the strict single-crossing condition, the modified types are ordered according to the weak single-crossing condition.[8]

Thus, to apply Theorem 1, there only remains to ensure that Assumption 1 holds for the marginal rates of substitution $\tau_i^T(q',0)$. A convenient way to proceed is to require that each type's family of primitive indifference curves satisfy a slightly stronger fanning-out condition than in Assumption 1.

Assumption 2 *For all* i *and* t, $\tau_i(q,t)$ *is nonincreasing in* q.

That is, a higher quantity traded reduces the buyer's willingness to pay. This assumption is satisfied by a large variety of preference relations, as we illustrate in Appendix C. In Appendix A, we establish the following result.

[6]Clearly, q in (5) should belong to the domain of T. The admissible set for q may also vary continuously in (q',t'), as for example when the consumption set of the buyer is bounded. To simplify notation, we do not explicitly mention such admissibility constraints in the maximization problems considered in this section.

[7]This follows from Berge's maximum theorem (Aliprantis and Border (2006, Theorem 17.31)).

[8]We refer to Attar, Mariotti, and Salanié (2019, Supplementary Appendix, Proof of Lemma 1) for a proof of the second and third observations.

Lemma 1 *If Assumption 2 holds for* $\tau_i(q,t)$*, then Assumption 1 holds for* $\tau_i^T(q',0)$.

We can now deduce from Theorem 1 that a tariff T is entry-proof if and only if

$$\text{For each } i, \quad \tau_i^T(0,0) \leq \bar{c}_i. \tag{6}$$

To see what this abstract condition entails for the tariff T and the allocation $(q_i, T(q_i))_{i=1}^I$ it implements, recall from (5) that $\tau_i^T(0,0)$ is the supremum of the set of prices p such that

$$u_i(q_i, T(q_i)) = u_i^T(0,0) < \max\{u_i^T(q', pq') : q'\} = \max\{u_i(q+q', T(q)+pq') : q, q'\}.$$

Thus, according to (6), we have

$$\text{For each } i, \quad u_i(q_i, T(q_i)) \geq \max\{u_i(q+q', T(q)+\bar{c}_i q') : q, q'\}. \tag{7}$$

Fixing $q_0 \equiv 0$ and applying (7) to $q \in [q_{i-1}, q_i]$ and $q' = q_i - q$ yields

$$\text{For all } i \text{ and } q \in [q_{i-1}, q_i], \quad T(q_i) \leq T(q) + \bar{c}_i(q_i - q). \tag{8}$$

In particular, at $q = q_{i-1}$, we have

$$T(q_i) \leq T(q_{i-1}) + \bar{c}_i(q_i - q_{i-1}). \tag{9}$$

Now, rewriting the expected profit (4) as in (2), and imposing that the allocation $(q_i, T(q_i))_{i=1}^I$ be budget-feasible, we have

$$\sum_i \left(\sum_{j \geq i} m_j \right) [T(q_i) - T(q_{i-1}) - \bar{c}_i(q_i - q_{i-1})] \geq 0.$$

The only possibility is thus that the inequalities (9) hold as equalities,

$$\text{For each } i, \quad T(q_i) = T(q_{i-1}) + \bar{c}_i(q_i - q_{i-1}), \tag{10}$$

which, in turn, implies, according to (7),

$$\text{For each } i, \quad u_i(q_i, T(q_i)) = \max\{u_i(q_{i-1}+q', T(q_{i-1})+\bar{c}_i q') : q'\}. \tag{11}$$

Finally, because T is convex and satisfies both (8) and (10), it must be that T is affine with slope $\bar{c}_i$ over the interval $[q_{i-1}, q_i]$. The following theorem, a formal proof of which is provided in the main appendix, summarizes this discussion and states that the necessary conditions derived above are also sufficient.

Theorem 2 *An allocation* $(q_i, T(q_i))_{i=1}^I$ *is budget-feasible and is implemented by an entry-*

proof convex market tariff T with domain $[0, q_I]$ if and only if they jointly satisfy the following recursive system:

(i) $(q_0, T(q_0)) \equiv (0, 0)$.

(ii) For each i, $q_i - q_{i-1} \in \arg\max\{u_i(q_{i-1} + q', T(q_{i-1}) + \overline{c}_i q') : q'\}$.

(iii) For each i, if $q_{i-1} < q_i$, then T is affine with slope $\overline{c}_i$ over the interval $[q_{i-1}, q_i]$.

In particular, any such allocation is exactly budget-balanced.

Let us first comment on each item of this result. First, it is natural to focus on tariffs defined up to the maximum quantity q_I—one can build other entry-proofs tariffs by suitably prolonging T beyond this point, but this is in no way needed. Next, (i) is merely a convention. Finally, (ii)–(iii) are substantial, and indicate how to recursively build a complete family of quantities, as well as a tariff that is by construction convex, because the upper-tail conditional expectation of unit costs is nondecreasing in the buyer's type.

Existence of an entry-proof convex market tariff obtains as soon as each maximization problem in (ii) admits a solution. This is ensured for example by the following Inada condition, which states that demand is finite when the price is positive:

$$\text{For all } i,\ (q, t), \text{ and } p > 0,\ \arg\max\{u_i(q + q', t + pq') : q'\} < \infty. \tag{12}$$

Therefore, under nonexclusivity, budget-feasibility and entry-proofness are not conflicting requirements, in contrast with the pervasive nonexistence problems arising under exclusivity (Rothschild and Stiglitz (1976)). We will return to this point in Section 6.

Uniqueness of an entry-proof convex market tariff also follows if the solution to each maximization problem in (ii) is unique. This is the case if the buyer's preferences are strictly convex. If they are only weakly convex, multiple solutions may appear if the marginal rate of substitution of some type i equals $\overline{c}_i$ over a whole interval of quantities, but this is clearly a nongeneric phenomenon.

Theorem 2 thus characterizes an essentially unique allocation. Following Attar, Mariotti, and Salanié (2014, 2019), we label this allocation, which was originally introduced in different contexts by Jaynes (1978), Hellwig (1988), and Glosten (1994), the *JHG allocation*, and we denote it by $(Q_i, T_i)_{i=1}^{I}$. Similarly, the *JHG tariff* consists of a sequence of layers with unit prices $\overline{c}_i$, and features an upward kink at any quantity $Q_i \in (0, Q_I)$ such that $Q_{i+1} > Q_i$ and $\overline{c}_{i+1} > \overline{c}_i$. In the limit-order book interpretation, this sequence of layers corresponds to a family of limit orders with maximum quantities $Q_i - Q_{i-1}$ and unit prices $\overline{c}_i$. The

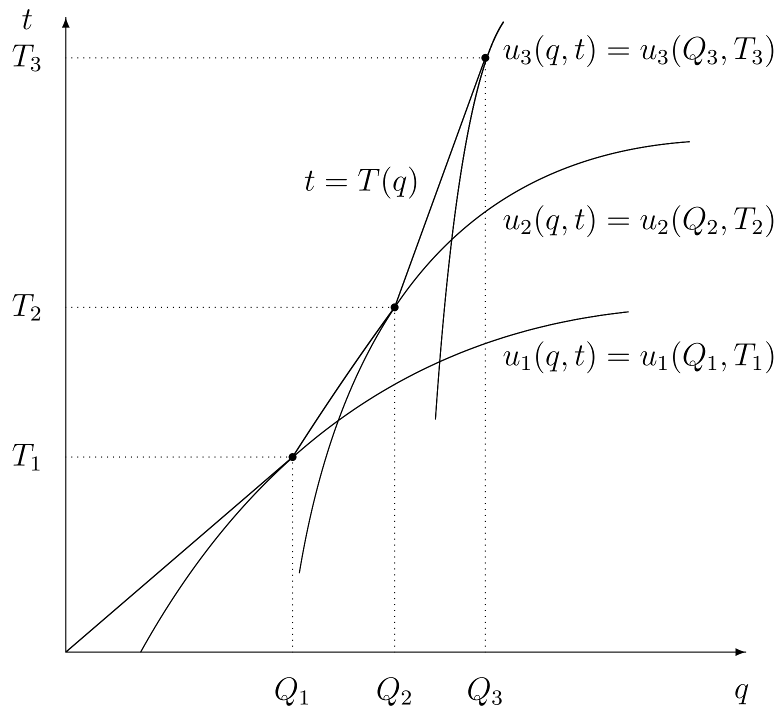

Figure 2: The JHG allocation and the JHG tariff for $I = 3$.

JHG allocation is exactly budget-balanced, because any marginal quantity is priced at the expected cost of serving the types who purchase it. This property can be interpreted as a marginal version of Akerlof (1970) pricing.

The JHG allocation and the JHG tariff are illustrated in Figure 2, in a case with three types and strictly convex preferences.

We can also apply Theorem 2 to preferences that are only weakly convex. For example, consider linear utility functions $u_i(q,t) \equiv v_i q - t$, subject to a capacity constraint $q \in [0,1]$. Such preferences generalize those in Akerlof (1970) to a divisible good. Strict single-crossing here requires that v_i be strictly increasing in i. Then each problem in (ii) admits a unique solution as soon as $v_i \neq \overline{c}_i$ for all i, and we have two possibilities:

1. If $v_i < \overline{c}_i$ for all i, then, according to (ii), all quantities must be zero. Moreover, by (iii), the essentially unique entry-proof tariff is only defined at zero, with $T(0) = 0$, and the market is inactive.

2. Suppose, alternatively, that $v_i > \overline{c}_i$ for some i, and let i^* be the lowest such i. Then, according to (ii), i^* trades up to capacity at unit price $\overline{c}_{i^*}$. By strict single-crossing, so do types $i > i^*$, while types $i < i^*$ do not trade at all. Moreover, by (iii), the unique entry-proof tariff is linear, with $T(q) = \overline{c}_{i^*} q$ for all $q \in [0,1]$.

Thus, generically, the JHG allocation features a single layer when the buyer's preferences are linear, and corresponds to the Akerlof (1970) competitive-equilibrium allocation that maximizes the gains from trade.

5 A Discriminatory Ascending Auction

Our results so far illustrate the power of the entry-proofness requirement, which selects an essentially unique market tariff and allocation. These elements in turn can be decomposed into successive layers, each of them priced at the expected cost of serving those types who trade it. As argued in the Introduction, it is natural to represent the JHG allocation as implemented via a dynamic process whereby trade first takes place at a low price until sellers stop serving the demand at this price, after which the price moves up, making sellers willing to supply additional quantities—and so on, until demand vanishes. We formalize this intuition by considering a discriminatory auction in which prices are quoted in ascending order. Each time a new price is quoted, each seller publicly announces the maximum quantity he stands ready to trade at this price, a process that can be interpreted as the sequential building of a limit-order book. Once this auctioning phase is completed, the buyer decides which quantities to purchase from which sellers in a nonexclusive way. This auction departs from the standard *tâtonnement* process in that sellers cannot withdraw the quantities they supplied at lower prices. As a consequence, the auction sequentially discovers a nonlinear market tariff instead of converging to a single equilibrium price. By construction, this tariff is convex, reflecting that the best price offers are optimally selected first.

5.1 Timing and Assumptions

We throughout postulate a discrete price grid. This is first for the sake of realism, as prices quoted on financial markets come in multiples of a minimum tick size. Second, an ascending auction with a discrete price grid can be modelled as a standard extensive-form game, allowing us to eschew the conceptual difficulties associated with continuous-time games (Simon and Stinchcombe (1989)). For a tick size $\Delta > 0$, we thus fix a price grid $\{0, \Delta, 2\Delta, \ldots\}$ and, to simplify the analysis, we assume that the upper-tail conditional expectations of unit costs $\overline{c}_i$ all belong to that grid. The game unfolds as follows.

In a first phase, the auctioneer quotes the prices in the grid in ascending order. When a new price p is quoted, $K \geq 2$ sellers simultaneously announce the maximum quantities $s^k(p) \geq 0$, $k = 1, \ldots, K$, they stand ready to trade at this price, which are publicly observable. The auctioneer then moves to the next price $p + \Delta$, and this process is repeated until all prices have been quoted.[9] Once this first phase is over, we can build a convex market

[9]We do not need to specify a stopping rule for this phase, because our game is formally well defined

tariff by aggregating the quantities successively supplied, as follows. Let $s(p) \equiv \sum_k s^k(p)$ be the aggregate supply at price p and $S(p) \equiv \sum_{p' \leq p} s(p)$ be the aggregate supply at prices lower than or equal to p. Then the tariff T is defined recursively by $T(0) \equiv 0$ and

$$\text{For each } Q \in [S(p-\Delta), S(p)], \quad T(Q) \equiv T(S(p-\Delta)) + p[Q - S(p-\Delta)].$$

In a second phase, the buyer learns her type, and decides which quantities to buy from which sellers. In the aggregate, she purchases a quantity Q in exchange for a payment $T(Q)$. Therefore, the price p of the last purchased unit is the left-derivative $\partial^- T(Q)$ of T at Q. The revenue earned by every seller k at any inframarginal price $p' < p$ is $p's^k(p')$, as it is in the buyer's interest to exhaust supply at any such price. The aggregate revenues earned by the sellers at price p are $p[Q - S(p-\Delta)]$. If $Q < S(p)$, the buyer is indifferent to the manner she allocates this revenue among the sellers; her equilibrium strategy will specify how she breaks these ties. Overall, a seller's expected profit is the expected sum of revenues at all prices, minus the expected cost of sales.

To simplify the exposition, we assume that every type i has quasilinear, strictly convex, and differentiable preferences satisfying the Inada condition (12), so that her demand $D_i(p)$ at any price $p > 0$ is single-valued, finite, and continuous and strictly decreasing in p as long as it is strictly positive. In particular, $D_i(p)$ goes to zero as p goes to ∞. Finally, to avoid nongeneric cases, we slightly strengthen the strict single-crossing condition by requiring that $D_i(p)$ be strictly increasing in i for each $p > 0$ as long as it is strictly positive.

We denote by Γ the corresponding extensive-form game with almost-perfect information. Our equilibrium concept is pure-strategy subgame-perfect Nash equilibrium. The remainder of this section provides our characterization results.

5.2 A Simple Equilibrium

In our equilibrium construction, the sellers' supply decisions at any history in the first phase of Γ only depend on the current price p and on the aggregate quantity Q^- supplied at prices $p' < p$. We call (p, Q^-) the current *state* of the game, which starts in state $(0, 0)$. In any state (p, Q^-), every type j has a residual demand $[D_j(p) - Q^-]^+$, where $[x]^+$ is the positive part of x. Under strict single-crossing, any quantity purchased at price p by some type i is also purchased by types $j > i$. Thus, in any state (p, Q^-) such that $\bar{c}_i < p \leq \bar{c}_{i+1}$, maximizing aggregate expected profits exactly requires serving the residual demand $[D_i(p) - Q^-]^+$ of type i, which we call the *profitable residual demand* in state (p, Q^-). The following theorem,

even with infinitely many prices in the grid. In practice, one may end this phase when the aggregate supply exceeds the highest possible demand at the current price. See also Footnote 10 for an alternative timing.

a formal proof of which is provided in the main appendix, exhibits an equilibrium of Γ in which sellers equally share this profitable residual demand in any state.

Theorem 3 *There exists an equilibrium of* Γ *in which, in any state* (p, Q^-),

(i) If $p \leq \overline{c}_1$*, each seller supplies a zero quantity.*

(ii) If $\overline{c}_1 < p \leq \overline{c}_I$*, each seller supplies an equal share of the profitable residual demand.*

(iii) If $p > \overline{c}_I$*, each seller supplies an infinite quantity.*

These strategies induce the following equilibrium outcome. As soon as the price reaches $\overline{c}_1 + \Delta$, the sellers collectively serve the demand $D_1(\overline{c}_1 + \Delta)$ of type 1, thereby satiating her demand; this quantity will also be purchased by types $i > 1$. Then, as soon as the price reaches $\overline{c}_2 + \Delta$, the sellers collectively serve the residual demand $[D_2(\overline{c}_2 + \Delta) - D_1(\overline{c}_1 + \Delta)]^+$ of type 2, thereby satiating her demand; this quantity will also be purchased by types $i > 2$. This process is repeated until the price reaches $\overline{c}_I + \Delta$, at which point the sellers flood the market by supplying an infinite quantity. It is readily checked that the resulting aggregate equilibrium allocation converges to the JHG allocation when Δ goes to zero. We will establish a general version of this result in the next section.

The proof of Theorem 3 relies on three arguments.

First, the game effectively stops when the price $\overline{c}_I + \Delta$ is reached. This allows us to apply the one-shot deviation property when analyzing the sellers' deviations.

Second, a seller may try to increase his market share $1/K$ in state (p, Q^-) by increasing his supply. Given his competitors' equilibrium strategies, however, it is easily seen that all profitable types at price p, that is, all types i such that $p > \overline{c}_i$, can choose to ignore this deviation and carry on trading the same quantity with each seller. Hence, the deviating seller will only succeed at selling more to nonprofitable types, which lowers his expected profit at price p. Moreover, because nonprofitable types trade more at this price, their residual demands at higher prices will also be reduced. Evaluating the overall impact on the continuation path, we show that such upward deviations cannot be profitable.

Third, a seller may try to reduce his supply in state (p, Q^-) by an amount q, so that some profitable type i is now rationed in this state. Her residual demand at the next price $p+\Delta$ should thus increase, say—for simplicity—by exactly q. The difficulty for the deviating seller is that the main part of this increase would go to his competitors. Indeed, following their equilibrium strategies, they would together react by supplying an additional amount $(K-1)q/K$ at price $p+\Delta$, leaving only q/K to him. Instead of selling q at price p, he would thus end up selling only q/K at price $p + \Delta$, which is less profitable as $\Delta < p$ and $K \geq 2$. Though this intuition is simple, the proof is more involved as demand is elastic. Hence the reduction in supply at the current price does not translate into an equivalent increase of the residual demand at the next price. Evaluating again the overall impact on the continuation path, we show that such downward deviations cannot be profitable.

Key to this existence result is the sequential nature of the ascending auction: at each price, each seller can condition his behavior on his competitors' past supply decisions. The only constraint is subgame-perfection, but this constraint is mild, as punishments take the form of profitable increases in supply. By contrast, simultaneous models of nonexclusive competition under adverse selection generally conclude to the nonexistence of equilibrium when preferences are strictly convex (Attar, Mariotti, and Salanié (2014, 2019)). Indeed, in such games, the natural candidate for equilibrium is similar to the one described in Theorem 3: each seller supplies a share of the profitable residual demand at each price and, therefore, is indispensable for serving that demand. The difference is that, in a simultaneous game, a seller can reduce his supply at a given price without triggering a reaction by his competitors. Indeed, the only available device to block such a deviation consists in the buyer sending appropriate reports to the nondeviating sellers, translating into different choices in the menus or tariffs they offer. Such reports, however, have to be sequentially rational from the buyer's viewpoint, which considerably restricts the set of available punishments. By contrast, in our equilibrium construction for the discriminatory ascending auction, the main thrust of punishments is borne by the sellers themselves, leaving for the buyer only the task of breaking ties at the expense of the deviating seller.

5.3 Convergence of Equilibrium Allocations

We now show that the JHG allocation uniquely emerges as the limit of equilibrium allocations when the tick size goes to zero, generalizing an insight of Theorem 3. To establish this result, we focus on buyers' equilibrium strategies that satisfy a minimal robustness requirement. Recall that, in equilibrium, every type i accepts all offers up to some price p_i. At this last price, the sellers' aggregate supply $s(p_i)$ may exceed her residual demand, so that type i can allocate it in different ways among the sellers. Although she is indifferent between all such allocations, her choice may matter to the sellers. We say that an equilibrium of Γ is *robust to irrelevant offers* if any type i's allocation of trades at price p_i does not depend on offers made at prices $p > p_i$. Intuitively, we do not allow the buyer to punish a seller for deviating at a price that is irrelevant to her as she is not willing to trade at this price. The equilibrium constructed in Theorem 3 satisfies this refinement.[10] The following theorem, a formal proof of which is provided in the main appendix, encapsulates our convergence result.

Theorem 4 *For each $n \in \mathbb{N}$, fix any equilibrium robust to irrelevant offers of the ascending auction Γ_n with tick size $\Delta_n \equiv \Delta/2^n$. Then the resulting sequence of aggregate equilibrium allocations converges to the JHG allocation.*

This result confirms the prominent role played by the JHG allocation under adverse selection and nonexclusive competition. Although the proof of Theorem 4 is involved, its logic follows from a generalized Bertrand argument that we sketch here, before turning to technical difficulties. To this end, let us hypothetically place ourselves in the limiting case $\Delta = 0$. Recall that, in any state (p, Q^-), type i's residual demand is $[D_i(p) - Q^-]^+$. When the aggregate supply in this state is s, sellers collectively earn

$$B(p, s, Q^-) \equiv \sum_i m_i(p - c_i) \min\{[D_i(p) - Q^-]^+, s\}.$$

Now, suppose, by way of contradiction, that there exists a state (p, Q^-) reached on the equilibrium path such that

$$B^*(p, Q^-) \equiv \max\{B(p, s, Q^-) : s \geq 0\} > 0. \tag{13}$$

Because the highest price at which trade takes place turns out to be bounded along the sequence of equilibria under consideration, let us, for the sake of the argument, focus on the highest price p satisfying (13). At even higher prices, B^* is at most zero. Notice, however, that aggregate continuation profits beyond p must be nonnegative; otherwise, some seller would find it profitable to stop making offers, without jeopardizing his profits at lower prices as the equilibrium is robust to irrelevant offers. Because aggregate continuation profits are an integral of aggregate expected profits B at all prices $p' > p$, each lying by construction below B^*, one must have $B = B^* = 0$ at any such price.

Now, at price p, sellers collectively earn at most $B^*(p, Q^-) > 0$. As a result, each seller is tempted to appropriate the totality of these aggregate expected profits. The classical Bertrand undercutting deviation consists in making a well-chosen offer at a price p' arbitrarily

[10] One simple manner to justify this refinement is to consider a slightly different timing for the game, as follows. At any price p quoted by the auctioneer, the sellers announce their supplies $s^k(p)$, and the buyer immediately reacts by choosing which quantities to purchase from which sellers. The game stops at price p if the buyer purchases less than the aggregate supply $s(p)$ at this price. Otherwise, the auctioneer goes to the next price $p + \Delta$—and so on. It is easily seen that in any perfect Bayesian equilibrium of this game, every type i optimally selects quantities by accepting all offers below some threshold p_i. At that threshold price, the game stops, so that the allocation of trades cannot depend on offers that will never be made.

close but below p at which the aggregate supply of his competitors is zero; such a price always exists if $\Delta = 0$. The deviating seller is then certain to attract in priority the relevant types, and to secure himself an expected profit at price p' arbitrarily close to $B^*(p, Q^-)$. Because continuation profits beyond p are zero and profits at lower prices are unaffected, this deviation is thus profitable, a contradiction. Therefore, we can conclude that B^* must at most be zero in every state reached on the equilibrium path, and thus, reasoning as above, that $B = B^* = 0$ in any such state. We show that this property actually characterizes the JHG allocation and the JHG tariff, which concludes the proof of Theorem 4.

Nevertheless, the technical difficulties in this reasoning should not be discounted. First, we need to establish a clear convergence result for supply functions and market tariffs when the tick size goes to zero. To do so, we rely on Helly's selection theorem. Second, it may well be that, in the limit, there exists no highest price p such that (13) holds. This requires a careful limiting argument. Third, with a discrete price grid, the price p' used for undercutting has to be either p—but then the deviating seller may not be first on the buyer's priority list—or $p - \Delta$—but at this price other sellers may also supply positive quantities, once more making priority difficult to achieve. Fortunately, when the tick size goes to zero, the number of available prices just below p grows without bounds. This guarantees that the aggregate supply of nondeviating sellers becomes negligible almost everywhere in a left-neighborhood of p, which validates the informal argument given above.

6 Discussion

6.1 Nonexclusivity and Discriminatory Pricing

Nonexclusive contracting is a key feature of prominent insurance markets such as annuities, life insurance, and long-term care, of consumer and firm credit markets, and of most financial markets. Early work has led researchers to the common conception that nonexclusivity is best represented by assuming linear pricing. For instance, Pauly (1974) explicitly assumes that insurance companies post price schedules that are linear in the amount of coverage purchased, and Chiappori (2000) argues that, under nonexclusivity, consumers can linearize any nonlinear schedule by trading many small contracts with different insurance companies. Similarly, in the context of annuities, Sheshinski (2008), Hosseini (2015), and Rothschild (2015) assume that each annuity contract is traded at the same unit price, equal to the average longevity of subscribers, weighted by the amount of annuities they purchase. In a general-equilibrium context, Bisin and Gottardi (1999) argue that a minimal form of nonlinear pricing, in the form of a bid-ask spread, ensures the existence of a competitive equilibrium in an adverse-selection economy with nonexclusive contracting; yet they retain the assumption that prices are linear on each side of the market.

Our results instead suggest that the intuition that nonexclusive contracting necessarily leads to linear pricing is fundamentally misleading: when the buyer's preferences are strictly convex and there is adverse selection in the weak sense that $\bar{c}_i$ is strictly increasing in i, the unique entry-proof convex tariff is nonlinear and different types end up trading at different marginal prices. The only exceptions are when preferences are linear, as in the example discussed at the end of Section 4, or when values are private, in which case adverse selection is not an issue. Moreover, as the implementability of the JHG allocation via an ascending discriminatory auction suggests, nonlinear pricing is consistent with nonexclusivity in a fully strategic context. Conceptually, the key point is that nonexclusivity does not entail anonymity: though no seller can monitor the trades a given buyer makes with his competitors, he can monitor the trades she makes with him. As a result, sellers can restrict the maximum quantity they trade at any price, as in the discriminatory limit-order book, and this ability prevents buyers from linearizing the convex tariffs they offer.

6.2 Exclusivity and Single-Crossing

A natural question is whether our approach can be applied to the exclusive-competition case studied by Rothschild and Stiglitz (1976). As we now argue, the key difference with the nonexclusive-competition case studied in Section 4 is that, when trades on the market and with an entrant are mutually exclusive, the buyer's indirect utility functions no longer satisfy single-crossing.

To see this point, observe that, if type i selects the contract $(q_i, T(q_i))$ along an exclusive market tariff T, an entrant can always offer a cream-skimming contract $(q_i - \delta, T(q_i) - \varepsilon)$, where $\alpha > 0$ and $\varepsilon > 0$ are chosen so as to attract type i without attracting types $j > i$. Thus the entrant's ability to target types is drastically enhanced, as she need not worry about adverse selection; in particular, our approach no longer applies. This suggests that the pervasive nonexistence problems arising in exclusive-competition models may not be due to private information or entry-proofness per se, but rather to this violation of single-crossing—or, to put it more provocatively, to the fact that these models do not incorporate the full extent of adverse selection.

By contrast, under nonexclusivity, any contract (q', t') offered by an entrant that attracts type i also attracts types $j > i$ if the market tariff T is convex. As shown in Section 4, the technical reason is that the indirect utility functions u_i^T inherit single-crossing from the primitive utility functions u_i. Intuitively, this is because the new contract (q', t') globally impacts the tariff, so that higher quantities $q > q'$ can now be purchased in exchange for a lower transfer $T(q - q') + t'$, making them more attractive to higher types. One may in fact argue that this is the basic mechanism through which adverse selection impedes entry: the introduction of new contracts is deterred by the fear of attracting types who are more costly to serve. This is why, under nonexclusivity, the relevant measure of cost from an entrant's perspective is the upper-tail conditional expectation of unit costs $\overline{c}_i$, as in Akerlof (1970), and not the unit cost c_i itself.

6.3 Beyond Convexity

In the previous section, we have highlighted the key role single-crossing plays in our analysis. This property itself resulted from the combination of two assumptions: that types be ordered according to the strict single-crossing condition, and that the market tariff be convex. We now examine to which extent this second assumption can be relaxed.

To do so, we propose a more direct characterization of the budget-feasible allocations $(q_i, T(q_i))_{i=1}^I$ that are implemented by an *arbitrary* entry-proof market tariff T. Because types are ordered according to the strict single-crossing condition, the optimal quantities q_i remain nondecreasing in i. A careful reading of the proof of Theorem 2 then reveals that, supposing (7) to hold, we can conclude that $(q_i, T(q_i))_{i=1}^I$ must be the JHG allocation; indeed, from (7) on, the convexity of T is not required to derive the desired equalities (10)–(11). Thus, what is needed is to directly establish (7) in a parsimonious way.

The most intuitive path is to proceed by contradiction, as follows. Suppose that (7) does not hold for some i, and consider a solution (q, q') to the maximization problem in (7). Then an entrant can offer the contract (q', t') with $t' \equiv \overline{c}_i q' + \varepsilon$ for some small $\varepsilon > 0$. Let J be the set of types that are attracted by this contract; by construction, J contains type i. To reach a contradiction, we must find conditions ensuring that the contract (q', t') is profitable. We explore two avenues in turn.

A Condition on the Distribution of Costs The first avenue is as follows. The worst case for the entrant is when J maximizes the expected cost $\mathbf{E}[c_j \,|\, j \in J]$ under the constraint $i \in J$. If the distribution of costs is such that the worst case occurs when the contract (q', t') attracts all types $j \geq i$, then entry is profitable as $t' > \overline{c}_i q'$. This implies the following result.

Lemma 2 *If $c_i \leq \overline{c}_i \leq c_{i+1}$ for all i, then the only budget-feasible allocation implemented by* *an entry-proof market tariff is the JHG allocation.*

The assumption of Lemma 2 is twofold: the first inequalities are equivalent to (1), while the second inequalities ensure that the worst case occurs when all types $j \geq i$ are attracted. In the two-type case, both inequalities hold as soon as $c_1 \leq c_2$. This shows that the convexity requirement is not needed in this simple case.[11]

Corollary 2 *In the two-type case, the only budget-feasible allocation implemented by an entry-proof market tariff is the JHG allocation.*

By contrast, the assumption of Lemma 2 becomes quite restrictive when the number of types grows large: indeed, in the limit, the corresponding set of cost distributions reduces to the private-value case in which c_i does not depend on i.

A Condition on the Market Tariff The second avenue is as follows. Once the contract (q', t') is offered, every type j purchases an aggregate quantity Q'_j and, as types are ordered according to the strict single-crossing condition, the quantities Q'_j are nondecreasing in j. Now, consider two types in J, say, to fix ideas, types 1 and 3. Then $Q'_1 \geq q'$, $Q'_3 \geq q'$, and

$$T(Q'_1) > T(Q'_1 - q') + t' \quad \text{and} \quad T(Q'_3) > T(Q'_3 - q') + t'. \tag{14}$$

At this point, let us assume that $T(q) - T(q - q')$ is quasiconcave in $q \geq q'$. Then, because the intermediate type 2 purchases an aggregate quantity $Q'_2 \in [Q'_1, Q'_3]$, (14) implies

$$T(Q'_2) > T(Q'_2 - q') + t',$$

and thus type 2 is also attracted by the contract (q', t'). This shows that J is connected. Under (1), the worst connected set is $\{j : j \geq i\}$, with expected cost $\bar{c}_i$, and we once more obtain that entry is profitable as $t' > \bar{c}_i q'$. There only remains to find a condition on T ensuring that $T(q) - T(q - q')$ is quasiconcave in $q \geq q'$ for all q'. The following result provides such a condition, which allows for tariffs exhibiting quantity discounts.

Corollary 3 *The only budget-feasible allocation implemented by an entry-proof market tariff that is first convex and then concave is the JHG allocation.*

Thus entry-proofness per se selects a convex tariff in a large class of admissible tariffs. Overall, we used the convexity of the market tariff only to ensure that adverse selection is sufficiently severe. The results in this section show that we can significantly relax this

[11] This result also appears in Attar, Mariotti and Salanié (2020), with a different proof.

assumption without threatening the special status of the JHG allocation. To go further, one would have to envision tariffs such that an entrant can attract a nonconnected set of types with an associated expected cost exceeding the upper-tail conditional expectation of unit costs, so that entry would be deterred even though (7) does not hold. In light of the above, this seems implausible, but we must acknowledge that the general problem remains open.

7 Concluding Remarks

In this paper, we have provided a unified perspective on entry-proofness under adverse selection, which is relevant both for inactive markets and for active markets in which buyers cannot be prevented from making additional trades with an entrant. These two scenarios turn out to be intimately linked: indeed, the second one reduces to the first one when buyers' utilities are modified to incorporate their optimal trades along the market tariff. Our existence and uniqueness results suggest that entry-proofness is a simple and powerful way to characterize the competitive outcomes of nonexclusive markets.

The JHG allocation and the JHG tariff that implements it emerge as the extension of Akerlof (1970) pricing to a rich class of preferences. The JHG allocation can be decomposed into successive layers, each of them priced at the expected cost of serving those types who trade it. This particular structure has motivated the design of a discriminatory ascending auction. In contrast with the simultaneous competitive-screening games so far studied in the literature, which generally conclude to the nonexistence of equilibrium, this sequential auction essentially uniquely implements the JHG allocation. Beyond making a theoretical point, this result offers a useful complement to studies that advocate a transformation of continuous markets into batch auctions, so as to avoid inefficiencies linked to high-frequency trading (Budish, Cramton, and Shin (2015)).

Although an empirical illustration is beyond the scope of this paper, our results suggest new avenues for empirical work. In the context of insurance, nonexclusive markets have been so far investigated through the lens of exclusive-competition models, exploiting the observation that, under adverse selection, there should be a positive correlation between the coverage purchased by a consumer and her risk (Chiappori and Salanié (2000)).[12] An alternative approach in line with our analysis would be to exploit price and cost data to compare the price of successive layers of insurance to their average cost, as measured by the empirical loss frequency of the consumers who trade them. This approach would extend the

[12]See, for instance, Cawley and Philipson's (1999) on life insurance, Finkelstein and Poterba (2004) on annuities, and Finkelstein and McGarry (2006) on long-term care.

one proposed by Einav, Finkelstein, and Cullen (2010) to richer environments where firms offer insurance tariffs and consumers can combine different levels of coverage from different firms, and the one proposed by Hendren (2013) in the case of inactive markets. Estimates of upper-tail conditional expectations of unit costs should be a key variable for future tests of adverse selection in nonexclusive insurance markets.

Finally, it is fair to acknowledge a limitation of our analysis. Following a time-honored tradition initiated by Akerlof (1970), Pauly (1974), and Rothschild and Stiglitz (1976), we have assumed that the buyers' private information is one-dimensional and that their types are ordered according to a single-crossing condition. These restrictions stand in contrast with the important role of multi-dimensional private information documented in the recent empirical literature.[13] There are in comparison few theoretical analyses of this question, and they have so far focused on exclusive-contracting environments.[14] An important challenge for future research is thus to understand the impact of multi-dimensional private information on the functioning of nonexclusive markets. Our hope is that the general methodology developed in this paper will prove useful to this end.

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[14] See, for instance, Chiappori, Jullien, Salanié, and Salanié (2006), Azevedo and Gottlieb (2017), and Guerrieri and Shimer (2018).

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Appendix

Proof of Theorem 1. The proof consists of three steps.

Step 1 We first formulate the entrant's problem. According to the revelation and taxation principles, there is no loss of generality in letting the entrant offer a menu of contracts $\{(q_1, t_1), \ldots, (q_I, t_I)\}$ that is incentive-compatible:

$$\text{For all } i \text{ and } j,\ u_i(q_i, t_i) \geq u_i(q_j, t_j),$$

and individually rational:

$$\text{For each } i,\ u_i(q_i, t_i) \geq u_i(0, 0).$$

We claim that, for any such menu, the buyer has a best response with quantities that are nondecreasing in her type. Indeed, if i optimally trades (q_i, t_i) and $j > i$ optimally trades (q_j, t_j), then it must be that $u_i(q_i, t_i) \geq u_i(q_j, t_j)$ and $u_j(q_j, t_j) \geq u_j(q_i, t_i)$. Now, suppose that $q_i > q_j$. Because $i < j$, applying weak single-crossing to the first inequality yields $u_j(q_i, t_i) \geq u_j(q_j, t_j)$, which, along with the second inequality, implies $u_j(q_i, t_i) = u_j(q_j, t_j)$. So type j could optimally trade (q_i, t_i) as well.[15] The same reasoning applies to any such pair (i, j) for which quantities are strictly decreasing, which proves the claim.

Because we want entry to be profitable no matter the buyer's best response, we are thus allowed to add the monotonicity constraint that quantities q_i be nondecreasing in i to the entrant's profit-maximization problem. We can also relax this problem by focusing on the downward local constraints, that is, the downward local incentive-compatibility constraints of types $i > 1$ and the individual-rationality constraint of type $i = 1$. The entrant's expected profit is thus bounded above by

$$\max \left\{ \sum_i m_i(t_i - c_i q_i) : q_i \text{ is nondecreasing in } i \text{ and } u_i(q_i, t_i) \geq u_i(q_{i-1}, t_{i-1}) \text{ for all } i \right\},$$

where $(q_0, t_0) \equiv (0, 0)$. We call $\mathcal{P}$ this relaxed problem.

Step 2 We now prove that we can focus in $\mathcal{P}$ on menus with nonnegative transfers. Indeed,

[15] Assuming strict single-crossing would enable us to reach a contradiction at this point, so that any best response of the buyer would feature nondecreasing quantities.

suppose that a menu $\{(q_1,t_1),\dots,(q_I,t_I)\}$ satisfies all the constraints in $\mathcal{P}$, and is such that at least one type makes a strictly negative payment. Let i be the lowest such type. Then we can build a new menu by assigning (q_{i-1},t_{i-1}) to both types $i-1$ and i. We claim that this new menu satisfies all the constraints in $\mathcal{P}$. First, because the original menu displays nondecreasing quantities, so does the new menu. Second, the downward local constraint for type i is now an identity. Third, the downward local constraint for type $i+1$, if such type exists, now writes as $u_{i+1}(q_{i+1},t_{i+1}) \geq u_{i+1}(q_{i-1},t_{i-1})$, which follows from observing that the initial menu satisfies $u_{i+1}(q_{i+1},t_{i+1}) \geq u_{i+1}(q_i,t_i)$, $q_i \geq q_{i-1}$, and $u_i(q_i,t_i) \geq u_i(q_{i-1},t_{i-1})$, and from applying weak single-crossing to the last inequality. This proves the claim. The resulting variation in expected profit is, up to multiplication by m_i,

$$(t_{i-1} - c_i q_{i-1}) - (t_i - c_i q_i) = t_{i-1} - t_i + c_i(q_i - q_{i-1}),$$

which is strictly positive as $t_{i-1} \geq 0 > t_i$ by construction and $q_i \geq q_{i-1}$. It follows that the initial menu cannot be solution to $\mathcal{P}$. The entrant's expected profit is thus bounded above by the value of the problem $\mathcal{P}_+$ obtained by adding to $\mathcal{P}$ the constraints $t_i \geq 0$ for all i.

Step 3 Fix a menu $\{(q_1,t_1),\dots,(q_I,t_I)\}$ that satisfies all the constraints in $\mathcal{P}_+$ and, for any type i, consider the trade (q_{i-1},t_{i-1}). For $i=1$, we clearly have $u_i(q_{i-1},t_{i-1}) \geq u_i(0,0)$ as $(q_0,t_0) = (0,0)$. For $i > 1$, we know that type $i-1$ weakly prefers (q_{i-1},t_{i-1}) to $(0,0)$. By weak single-crossing, so does type i. Thus, in any case, we have $u_i(q_{i-1},t_{i-1}) \geq u_i(0,0)$. Because $t_{i-1} \geq 0$, this shows that the indifference curve of type i going through (q_{i-1},t_{i-1}) must cross the q-axis at some point $(\underline{q}_i, 0)$, with $\underline{q}_i \in [0, q_{i-1}]$. The argument in the text then shows that $\sum_i m_i(t_i - c_i q_i) \leq 0$. Hence the result. ■

Proof of Corollary 1. According to Footnote 15, strict single-crossing implies that any best response of the buyer features nondecreasing quantities. Suppose, by way of contradiction, that the entrant trades, so that $q_i > q_{i-1}$ for some type i. Because any such type's preferences are strictly convex and $u_i(q_i,t_i) \geq u_i(q_{i-1},t_{i-1})$, the inequalities (3) now imply

$$t_i - t_{i-1} - \overline{c}_i(q_i - q_{i-1}) < 0.$$

Thus the expected profit (2) is strictly negative, a contradiction. Hence the result. ■

Proof of Theorem 2. The necessity part is shown in the text. Assume now that (i)–(iii) hold. From (iii), T is defined over $[0, q_I]$, and it is convex because $\overline{c}_i$ is nondecreasing in i. The proof consists of two steps.

Step 1 We first check that T implements the quantities q_i, in the sense that, for each i,

q_i maximizes $u_i(q, T(q))$ with respect to q. This is easily shown by induction. First, from (ii), type 1 optimally chooses q_1 when facing the tariff $T_1(q) = \bar{c}_1 q$. Because $T_1 \leq T$ and $T_1(q_1) = T(q_1)$, it follows that q_1 is indeed an optimal choice for type 1 when facing T. Next, suppose that type $i-1$ optimally chooses q_{i-1} when facing T. By weak single-crossing, for type i we can then focus on quantities $q \geq q_{i-1}$. From (ii), type i optimally chooses q_i when facing the tariff T_i that coincides with T for quantities $q \leq q_{i-1}$ and has slope $\bar{c}_i$ beyond q_{i-1}. Because $T_i \leq T$ and $T_i(q_i) = T(q_i)$, it follows that q_i is indeed an optimal choice of type i when facing T. This concludes the induction step.

Step 2 To conclude the proof, we only need to check that (6) holds for the tariff T defined by (i)–(iii). For each i, $\tau_i^T(0,0)$ is the supremum of the prices p such that

$$u_i(q_i, T(q_i)) < \max\{u_i(q + q', T(q) + pq') : q, q'\} \equiv U_i^T(p). \tag{15}$$

Let us compute $U_i^T(\bar{c}_i)$. Because $\partial^- T(q) \leq \bar{c}_i$ for $q < q_{i-1}$ and $\partial^- T(q) \geq \bar{c}_i$ for $q > q_{i-1}$, there exists for $p = \bar{c}_i$ a solution to the maximization problem in (15) such that $q = q_{i-1}$. It then follows from (ii) that $U_i^T(\bar{c}_i) = u_i(q_i, T(q_i))$. Thus (15) does not hold for $p = \bar{c}_i$, which implies $\tau_i^T(0,0) \leq \bar{c}_i$ as $U_i^T(p)$ is nondecreasing in p. Hence the result. ■

Proof of Theorem 3. We throughout set $\min \emptyset \equiv \infty$ and $\sum_{j<0} = \sum_{j>I} \equiv 0$. The proof consists of two steps.

Step 1 We first compute each seller's continuation profit in state (p, Q^-), which is the sum of all the expected profits he earns at prices $p' \geq p$ by trading with every type i such that $D_i(p) > Q^-$. In any of her best responses, any such type purchases Q^- at prices $p' < p$.

Case 1: $p > \bar{c}_I$ According to (iii), for any value of Q^-, each seller supplies an infinite quantity in state (p, Q^-). The best response we select for every type i is to equally split her residual demand in state (p, Q^-) between the sellers. Because each type can purchase her demand at price p, she makes no purchases at prices $p' > p$. Thus each seller's continuation profit in state (p, Q^-) is

$$\sum_i m_i(p - c_i) \frac{[D_i(p) - Q^-]^+}{K}. \tag{16}$$

Case 2: $\bar{c}_i < p \leq \bar{c}_{i+1}$ According to (ii), for any value of Q^-, each seller supplies an equal share of type i's residual demand in state (p, Q^-). The best response we select for every type $j < i$ is to equally split her residual demand in state (p, Q^-) between the sellers. Next, by strict single-crossing, every type $j \geq i$ purchases $[D_i(p) - Q^-]^+/K$ from each seller at price p.

Finally, each seller earns a continuation profit, which is the sum of all his expected profits at prices $p' > p$. Because type i can purchase her demand at price p, she makes no purchases at prices $p' > p$. To characterize the types who make purchases in excess of $\max\{D_i(p), Q^-\}$, we rank the demands $\mathcal{D}_j \equiv D_j(\bar{c}_j + \Delta)$ according to the following recursive definition.

Definition 1 *Let* $r(1) \equiv 1$ *and, for each* ι, *let* $r(\iota+1) \equiv \min\{j : j > r(\iota)$ *and* $\mathcal{D}_j > \mathcal{D}_{r(\iota)}\}$. *Let* $\bar{\iota} \equiv \max\{\iota : r(\iota) < \infty\}$ *and* $\mathcal{D}_\infty \equiv \infty$.

Now, let $\iota_i(p, Q^-) \equiv \min\{\iota : r(\iota) > i$ and $\mathcal{D}_{r(\iota)} > \max\{D_i(p), Q^-\}\}$. According to (ii), $\bar{c}_{r(\iota_i(p,Q^-))} + \Delta$ is the first price at which a quantity in excess of $\max\{D_i(p), Q^-\}$ is supplied, and $r(\iota_i(p, Q^-))$ is the first type willing to purchase some of it. By strict single-crossing, every type $j \geq r(\iota_i(p, Q^-))$ purchases $(\mathcal{D}_{r(\iota_i(p,Q^-))} - \max\{D_i(p), Q^-\})/K$ from each seller at price $\bar{c}_{r(\iota_i(p,Q^-))} + \Delta$, so that the expected margin on these trades is Δ. Next, every type $j \geq r(\iota_i(p, Q^-) + 1)$ purchases $(\mathcal{D}_{r(\iota_i(p,Q^-)+1)} - \mathcal{D}_{r(\iota_i(p,Q^-))})/K$ from each seller at price $\bar{c}_{r(\iota_i(p,Q^-)+1)} + \Delta$—and so on. Thus each seller's continuation profit in state (p, Q^-) is

$$\begin{aligned} &\sum_{j<i} m_j (p - c_j) \frac{[D_j(p) - Q^-]^+}{K} \\ &+ \left(\sum_{j \geq i} m_j\right)(p - \bar{c}_i) \frac{[D_i(p) - Q^-]^+}{K} \\ &+ \left(\sum_{j \geq r(\iota_i(p,Q^-))} m_j\right) \Delta \frac{\mathcal{D}_{r(\iota_i(p,Q^-))} - \max\{D_i(p), Q^-\}}{K} \\ &+ \sum_{\iota=\iota_i(p,Q^-)+1}^{\bar{\iota}} \left(\sum_{j \geq r(\iota)} m_j\right) \Delta \frac{\mathcal{D}_{r(\iota)} - \mathcal{D}_{r(\iota-1)}}{K}. \end{aligned} \tag{17}$$

Case 3: $p \leq \bar{c}_1$ Let $\iota_0(Q^-) \equiv \min\{\iota : \mathcal{D}_{r(\iota)} > Q^-\}$. According to (i)–(ii), $\bar{c}_{r(\iota_0(Q^-))} + \Delta$ is the first price at which a quantity in excess of Q^- is supplied, and $r(\iota_0(Q^-))$ is the first type willing to purchase some of it. Thus each seller's continuation profit in state (p, Q^-) is

$$\left(\sum_{j \geq r(\iota_0(Q^-))} m_j\right) \Delta \frac{\mathcal{D}_{r(\iota_0(Q^-))} - Q^-}{K} + \sum_{\iota=\iota_0(Q^-)+1}^{\bar{\iota}} \left(\sum_{j \geq r(\iota)} m_j\right) \Delta \frac{\mathcal{D}_{r(\iota)} - \mathcal{D}_{r(\iota-1)}}{K}.$$

Step 2 We now check that no seller can strictly increase his continuation profit by deviating from the candidate-equilibrium strategy. As the buyer's decisions to purchase from each seller at prices $p' < p$ do not depend on the offers he makes at prices $p' \geq p$, this implies that no deviation is profitable.

Case 1: $p > \bar{c}_I$ According to (iii), for any value of Q^-, each seller supplies an infinite quantity in state (p, Q^-). If a seller deviates to a finite q, then the best response we select for every type i is to purchase $\min\{q, [D_i(p) - Q^-]^+/K\}$ from him at price p. Because $p > \bar{c}_I \geq c_i$ for all i, (16) implies that the deviating seller cannot thereby strictly increase his continuation profit. Thus no seller has an incentive to deviate, and the game ends in state $(p + \Delta, \infty)$. In particular, whatever the sellers' decisions at prices $p' \leq \bar{c}_I$, the highest price at which trade can take place is $\bar{c}_I + \Delta$. This allows us to apply the one-shot deviation property at prices $p' \leq \bar{c}_I$.

Case 2: $\bar{c}_i < p \leq \bar{c}_{i+1}$ According to (ii), for any value of Q^-, each seller supplies an equal share of type i's residual demand in state (p, Q^-). If a seller deviates to a quantity q, the aggregate supply at prices $p' \leq p$ becomes

$$S(p, Q^-, q) \equiv Q^- + \frac{K-1}{K} [D_i(p) - Q^-]^+ + q. \tag{18}$$

We consider two types of deviations in turn.

Downward Deviations If $D_i(p) > Q^-$, a seller can deviate to $q < [D_i(p) - Q^-]/K$. We compute his continuation profit from doing so by using the one-shot deviation property. First, the best response we select for every type $j < i$ is to purchase $\min\{q, [D_j(p) - Q^-]^+/K\}$ from him at price p. Next, type i is rationed at price p because the aggregate supply at prices $p' \leq p$ is $S(p, Q^-, q) < D_i(p)$ by (18). Hence, by strict single-crossing, every type $j \geq i$ purchases q from the deviating seller at price p. Finally, the deviating seller earns a continuation profit, which is the sum of all his expected profits at prices $p' > p$ and can be computed as in (17), with p replaced by $p + \Delta$ and Q^- replaced by $S(p, Q^-, q)$. Thus each seller's continuation profit from deviating to $q < [D_i(p) - Q^-]/K$ in state (p, Q^-) and returning to equilibrium play afterwards is

$$\begin{aligned}
\sum_{j<i} m_j \left[(p - c_j) \min\left\{ q, \frac{[D_j(p) - Q^-]^+}{K} \right\} + (p + \Delta - c_j) \, \frac{[D_j(p+\Delta) - S(p, Q^-, q)]^+}{K} \right] \\
+ \left(\sum_{j \geq i} m_j \right) (p - \bar{c}_i) q \\
+ \left(\sum_{j \geq i} m_j \right) (p + \Delta - \bar{c}_i) \, \frac{[D_i(p+\Delta) - S(p, Q^-, q)]^+}{K} \\
+ \left(\sum_{j \geq r(\iota_i(p+\Delta, S(p, Q^-, q)))} m_j \right) \Delta \, \frac{\mathcal{D}_{r(\iota_i(p+\Delta, S(p, Q^-, q)))} - \max\{D_i(p+\Delta), S(p, Q^-, q)\}}{K}
\end{aligned}$$

$$+\sum_{\iota=\iota_i(p+\Delta,S(p,Q^-,q))+1}^{\bar\iota}\left(\sum_{j\geq r(\iota)} m_j\right)\Delta\,\frac{\mathcal{D}_{r(\iota)}-\mathcal{D}_{r(\iota-1)}}{K}. \quad (19)$$

To compare this to (17), we use the definition (18) of $S(p,Q^-,q)$. As $D_j(p+\Delta) > S(p,Q^-,q)$ implies $D_j(p) > Q^-$, we first obtain that the coefficient of q in each term of the first sum in (19), when different from zero, is at least

$$(p-c_j)\left(1-\frac{1}{K}\right)-\frac{\Delta}{K}\geq\left(1-\frac{2}{K}\right)\Delta\geq 0$$

because $p\geq\bar c_i+\Delta\geq c_j+\Delta$ for $j<i$, and $K\geq 2$. Similarly, by distinguishing whether $D_i(p+\Delta)$ is higher or lower than $S(p,Q^-,q)$, we obtain that the coefficient of q in the next three terms in (19) is at least

$$\left(\sum_{j\geq i} m_j\right)\left[(p-\bar c_i)\left(1-\frac{1}{K}\right)-\frac{\Delta}{K}\right]\geq\left(\sum_{j\geq i} m_j\right)\left(1-\frac{2}{K}\right)\Delta\geq 0$$

because $p\geq\bar c_i+\Delta$ and $K\geq 2$. Hence choosing $q'\in(q,[D_i(p)-Q^-]/K]$ instead of q never decreases the deviating seller's continuation profit as long as $\iota_i(p+\Delta,S(p,Q^-,q'))$ remains constant. Eventually, however, this index may jump up, in which case the last sum in (19) jumps down. When q' is close to but below the value at which such a jump occurs, then $\max\{D_i(p+\Delta),S(p,Q^-,q')\}=S(p,Q^-,q')$ becomes close to $\mathcal{D}_{r(\iota_i(p+\Delta,S(p,Q^-,q')))}$, and hence the third and fourth terms in (19) vanish while the second term in (19) becomes close to

$$\left(\sum_{j\geq i} m_j\right)(p-\bar c_i)\left\{\mathcal{D}_{r(\iota_i(p+\Delta,S(p,Q^-,q')))}-Q^--\frac{K-1}{K}\,[D_i(p)-Q^-]\right\}.$$

As the first sum in (19) is at most equal to the first sum in (17), this reasoning shows that all we need to prove is that $\pi\geq\pi(\hat\iota)$ for all $\hat\iota=\iota_i(p+\Delta,S(p,Q^-,q)),\ldots,\iota_i(p,Q^-)-1$, where

$$\pi\equiv\left(\sum_{j\geq i} m_j\right)(p-\bar c_i)\,\frac{D_i(p)-Q^-}{K}$$
$$+\left(\sum_{j\geq r(\iota_i(p,Q^-))} m_j\right)\Delta\,\frac{\mathcal{D}_{r(\iota_i(p,Q^-))}-D_i(p)}{K}+\sum_{\iota=\iota_i(p,Q^-)+1}^{\bar\iota}\left(\sum_{j\geq r(\iota)} m_j\right)\Delta\,\frac{\mathcal{D}_{r(\iota)}-\mathcal{D}_{r(\iota-1)}}{K}$$

and, for any such $\hat\iota$,

$$\pi(\hat\iota)\equiv\left(\sum_{j\geq i} m_j\right)(p-\bar c_i)\left\{\mathcal{D}_{r(\hat\iota)}-Q^--\frac{K-1}{K}\,[D_i(p)-Q^-]\right\}$$
$$+\sum_{\iota=\hat\iota+1}^{\bar\iota}\left(\sum_{j\geq r(\iota)} m_j\right)\Delta\,\frac{\mathcal{D}_{r(\iota)}-\mathcal{D}_{r(\iota-1)}}{K}.$$

For each $\hat{\iota} = \iota_i(p+\Delta, S(p, Q^-, q)) + 1, \ldots, \iota_i(p, Q^-) - 1$, we have

$$\pi(\hat{\iota}) - \pi(\hat{\iota}-1) = \left[\left(\sum_{j \geq i} m_j\right)(p - \bar{c}_i) - \left(\sum_{j \geq r(\hat{\iota})} m_j\right)\frac{\Delta}{K}\right](\mathcal{D}_{r(\hat{\iota})} - \mathcal{D}_{r(\hat{\iota}-1)}),$$

which is strictly positive because $r(\hat{\iota}) > i$, $p \geq \bar{c}_i + \Delta$, $K \geq 2$, and $\mathcal{D}_{r(\hat{\iota})} > \mathcal{D}_{r(\hat{\iota}-1)}$. Hence, to conclude, we only need to check that $\pi \geq \pi(\iota_i(p, Q^-) - 1)$. We have

$$\pi - \pi(\iota_i(p, Q^-) - 1) = \left[\left(\sum_{j \geq i} m_j\right)(p - \bar{c}_i) - \left(\sum_{j \geq r(\iota_i(p,Q^-))} m_j\right)\frac{\Delta}{K}\right][D_i(p) - \mathcal{D}_{r(\iota_i(p,Q^-)-1)}],$$

which is nonnegative because $r(\iota_i(p, Q^-)) > i$, $p \geq \bar{c}_i + \Delta$, $K \geq 2$, and $D_i(p) \geq \mathcal{D}_{r(\iota_i(p,Q^-)-1)}$ by definition of $\iota_i(p, Q^-)$. This concludes the proof that no deviation to $q < [D_i(p) - Q^-]/K$ can increase a seller's continuation profit in state (p, Q^-).

Upward Deviations A seller can deviate to $q > [D_i(p) - Q^-]^+/K$. We compute his continuation profit from doing so by using the one-shot deviation property. First, the best response we select for every type $j \leq i$ is to purchase $[D_j(p) - Q^-]^+/K$ from him at price p. Next, the best response we select for every type $j > i$ is to purchase

$$\min\left\{q, [D_j(p) - Q^-]^+ - \frac{K-1}{K}[D_i(p) - Q^-]^+\right\} \geq \frac{1}{K}[D_i(p) - Q^-]^+ \tag{20}$$

from him at price p, reflecting that any such type can first purchase $[D_i(p) - Q^-]^+/K$ from each of the nondeviating sellers and then purchase any additional quantity she is willing to purchase at price p from the deviating seller, within the limit q. Finally, the deviating seller earns a continuation profit, which can be computed as in the case of downward deviations. Thus each seller's continuation profit from deviating to $q > [D_i(p) - Q^-]^+/K$ in state (p, Q^-) and returning to equilibrium play afterwards is

$$\begin{aligned}
&\sum_{j<i} m_j(p - c_j)\frac{[D_j(p) - Q^-]^+}{K} \\
&+ \sum_{j \geq i} m_j(p - c_j)\min\left\{q, [D_j(p) - Q^-]^+ - \frac{K-1}{K}[D_i(p) - Q^-]^+\right\} \\
&+ \left(\sum_{j \geq r(\iota_i(p,S(p,Q^-,q)))} m_j\right)\Delta\,\frac{\mathcal{D}_{r(\iota_i(p,S(p,Q^-,q)))} - S(p, Q^-, q)}{K} \\
&+ \sum_{\iota=\iota_i(p,S(p,Q^-,q))+1}^{\bar{\iota}}\left(\sum_{j \geq r(\iota)} m_j\right)\Delta\,\frac{\mathcal{D}_{r(\iota)} - \mathcal{D}_{r(\iota-1)}}{K}.
\end{aligned} \tag{21}$$

The first sum in (21) is the same as in (17). Next, using a summation by parts and (20), we obtain that the second sum in (21) is of the form

$$\sum_{j\geq i}\left(\sum_{k\geq j} m_k\right)(p-\bar{c}_j)(q_j-q_{j-1})$$

for nondecreasing quantities $(q_j)_{j=i-1}^{I}$ such that $q_{i-1}\equiv 0$ and $q_i \equiv [D_i(p)-Q^-]^+/K$. Because $p \leq \bar{c}_j$ for all $j > i$, this sum is at most equal to its first term corresponding to $j = i$, which itself is equal to the second term in (17). Finally, $S(p,Q^-,q) > \max\{D_i(p),Q^-\}$ and $\iota_i(p,S(p,Q^-,q)) \geq \iota_i(p,Q^-)$ imply that the last two terms of (21) are at most equal to the last two terms of (17). This concludes the proof that no deviation to $q > [D_i(p)-Q^-]^+/K$ can increase a seller's continuation profit in state (p,Q^-).

Case 3: $p \leq \bar{c}_1$ According to (i), for any value of Q^-, each seller supplies a zero quantity at price p. Thus no downward deviation is feasible. The proof that no upward deviation can increase a seller's continuation profit in state (p,Q^-) is similar to that provided in Case 2 and is thus omitted. Hence the result. ■

Proof of Theorem 4. Every type i's preferences can be represented by $U_i(q)-t$ for some strictly concave utility function U_i that is differentiable over $\mathbb{R}_{++}$. The Inada condition (12), which is here equivalent to $\lim_{q\to\infty} U_i'(q) \leq 0$, ensures that $D_i(p) < \infty$ except perhaps for $p = 0$. We will often use the property that, when facing a convex market tariff, each type optimally purchases the totality of the sellers' supply until her demand is satisfied at some price or, equivalently, until the price exceeds her willingness-to-pay. Therefore, if type i trades at price p, then she overall purchases at most $D_i(p)$; if she at least purchases $q > 0$, then she is not willing to trade at prices $p > U_i'(q)$.

We first dispose of the case where the JHG allocation $(Q_i,T_i)_{i=1}^{I}$ is degenerate, that is, $Q_I = 0$. Then, by Theorem 2, Condition EP is satisfied. As the buyer's preferences are strictly convex and types are ordered according to the strict single-crossing condition, it follows from Corollary 1 that there is market breakdown. Thus no trade takes place in any equilibrium of any game Γ_n, and each equilibrium implements the degenerate JHG allocation. Hence the result.

From now on, we assume that the JHG allocation is nondegenerate, that is, $Q_I > 0$. By Theorem 2, this amounts to assuming that there exists some i such that $U_i'(0) > \bar{c}_i$ or, equivalently, $D_i(\bar{c}_i) > 0$.

Our first task is to show that we can put uniform bounds on equilibrium prices and quantities. The proof of the following lemma—and of all the intermediary results used in the proof of Theorem 4—is provided in Appendix A.

Lemma 3 *There exist a finite price $\overline{p}$ and finite quantities $\overline{Q} > \underline{Q} > 0$ such that, for n high enough, in any equilibrium of Γ_n type I is not willing to trade at prices strictly higher than $\overline{p}$ and purchases an aggregate quantity in $[\underline{Q}, \overline{Q}]$.*

Thanks to this result, we can in what follows consider that the auction ends when price $\overline{p}$ is reached, that supply functions are defined over $[0, \overline{p}]$ and bounded above by $\overline{Q}$, and that tariffs are defined over $[0, \overline{Q}]$. This makes no difference for the quantities chosen by the buyer on the equilibrium path, and the profitability of the deviation we shall soon consider does not depend on the values of these functions at higher arguments. Hence, for n high enough and for any equilibrium of Γ_n, there exists a finite highest price $p_{i,n}$ at which type i trades on the equilibrium path; we set $p_{i,n} \equiv U_i'(0)$ if type i does not trade. By strict single-crossing, $p_{1,n} \leq p_{2,n} \leq \ldots \leq p_{I,n} < \overline{p}$. Let $Q_{i,n}$ be the aggregate quantity purchased by type i on the equilibrium path. By strict single-crossing again, $Q_{1,n} \leq Q_{2,n} \leq \ldots \leq Q_{I,n} \leq \overline{Q}$.

From now on, we fix a sequence of equilibria of $(\Gamma_n)_{n \in \mathbb{N}}$ that are robust to irrelevant offers. For each n, the following objects are defined on the equilibrium path:

- $s_n^k(p)$, seller k's supply at price p;
- $s_n(p) \equiv \sum_k s_n^k(p)$, the aggregate supply at price p;
- $s_n^{-k}(p) \equiv s_n(p) - s_n^k(p)$, the aggregate supply of sellers other than k at price p;
- $S_n(p) \equiv \sum_{p' \leq p} s_n(p')$, the aggregate supply at prices lower than or equal to p;
- $\pi_n^k(p)$, seller k's expected profit at price p;
- $\gamma_n^k(p) \equiv \sum_{p' \geq p} \pi_n^k(p')$, seller k's continuation profit at price p.

In any equilibrium of Γ_n that is robust to irrelevant offers, each seller anticipates that deviating at prices $p' \geq p$ will not affect the buyer's decisions at prices $p' < p$. We can thus focus on continuation profits as in the proof of Theorem 3. This, in particular, implies that $\gamma_n^k(p) \geq 0$ for all k and p; otherwise, seller k could strictly increase his expected profit by withdrawing his offers at prices $p' \geq p$.

To formulate our convergence result, we extend the supply functions $(S_n)_{n \in \mathbb{N}}$ to the whole of $[0, \overline{p}]$ by letting

$$\text{For all } n \text{ and } p, \quad S_n(p) \equiv S_n(\Delta_n \lfloor p/\Delta_n \rfloor),$$

where $\lfloor p/\Delta_n \rfloor$ is the integer part of p/Δ_n. By construction, for each n, the function S_n is nondecreasing and right-continuous; moreover, for each $p \in [0, \overline{p}]$ and for n high enough,

$S_n(p) \in [0, \overline{Q}]$. Therefore, by Helly's selection theorem (Billingsley (1995, Theorem 25.9)), there exists a nondecreasing right-continuous function S_∞ and a subsequence of $(S_n)_{n\in\mathbb{N}}$ that converges pointwise to S_∞ over $[0, \overline{p}]$ at the continuity points of S_∞. In what follows, and with no loss of generality, we take this subsequence to be the original sequence $(S_n)_{n\in\mathbb{N}}$. The marginal tariffs associated to S_n and S_∞ are their generalized inverses

$$\text{For each } q \in [0, \overline{Q}], \quad t_n(Q) \equiv \inf\{p : Q \leq S_n(p)\} \quad \text{and} \quad t_\infty(Q) \equiv \inf\{p : Q \leq S_\infty(p)\},$$

with $\inf \emptyset \equiv \overline{p}$; they are nondecreasing and left-continuous. It follows from the proof of Skorokhod's representation theorem (Billingsley (1995, Theorem 25.6)) that the sequence $(t_n)_{n\in\mathbb{N}}$ converges pointwise to t_∞ at the continuity points of t_∞, that is, everywhere over $[0, \overline{Q}]$ except at countably many points. Letting T_n and T_∞ be the convex tariffs obtained by integrating the marginal tariffs t_n and t_∞, we then have

$$\sup_{Q\in[0,\overline{Q}]} |T_n(Q) - T_\infty(Q)| = \sup_{Q\in[0,\overline{Q}]} \left| \int_0^Q [t_n(q) - t_\infty(q)] \, \mathrm{d}q \right| \leq \int_0^{\overline{Q}} |t_n(q) - t_\infty(q)| \, \mathrm{d}q,$$

which converges to zero by the bounded convergence theorem as the functions $(t_n)_{n\in\mathbb{N}}$ are uniformly bounded by $\overline{p}$ and converge pointwise to t_∞ except at countably many points. Thus the sequence $(T_n)_{n\in\mathbb{N}}$ converges uniformly to T_∞. This implies that the graph of T_∞ is the closed limit of the graph of T_n as n goes to ∞ (Aliprantis and Border (2006, Definition 3.80)). As a result, and because every type i has strictly convex preferences, we can conclude from Berge's maximum theorem (Aliprantis and Border (2006, Theorem 17.31)) that $Q_{i,\infty} \equiv \lim_{n\to\infty} Q_{i,n}$ is well defined and is the unique optimal choice of type i against the limit tariff T_∞. By Lemma 3, we have $Q_{I,\infty} \geq \underline{Q} > 0$.

With these preliminaries at hand, we turn to our main argument. As $\lim_{n\to\infty} Q_{i,n} = Q_{i,\infty}$ and the sequence $(T_n)_{n\in\mathbb{N}}$ converges uniformly to T_∞, $\lim_{n\to\infty} T_n(Q_{i,n}) = T_\infty(Q_{i,\infty})$. Our goal is to show that $(Q_{i,\infty}, T_\infty(Q_{i,\infty}))_{i=1}^I$ is the JHG allocation. We will rely on the following characterization of the JHG allocation, which is of independent interest.

Lemma 4 *The allocation implemented by a convex tariff T is the JHG allocation if and only if it is budget-feasible and*

$$\text{For all } p \text{ and } s, \quad B(p, s) \equiv \sum_i m_i(p - c_i) \min\{[D_i(p) - S(p^-)]^+, s\} \leq 0, \tag{22}$$

where S is the supply function associated to T and $S(p^-) \equiv \lim_{p'\uparrow p} S(p')$.

We denote by B_n and B_∞ the functions B in (22) obtained for $S = S_n$ and $S = S_\infty$, respectively. A key observation is that the functions π_n^k, B_n, and s_n are related as follows:

$$\text{For each } p, \quad \sum_k \pi_n^k(p) = B_n(p, s_n(p)). \tag{23}$$

The allocation $(Q_{i,\infty}, T_\infty(Q_{i,\infty}))_{i=1}^I$ is budget-balanced as it is the limit of the equilibrium allocations. Thus, by Lemma 4, to prove that it coincides with the JHG allocation, we only need to show that (22) holds for B_∞. Thus suppose, by way of contradiction, that there exists some p such that $B^*_\infty(p) \equiv \max\{B_\infty(p,s) : s \geq 0\} > 0$, and let $\hat{p}_\infty$ the supremum of such p. Our next result gathers useful properties related to this threshold.

Lemma 5 *The following holds:*

(i) $\hat{p}_\infty \leq \overline{p}$.

(ii) $B^*_\infty(p) > 0$ *if and only if there exists some* i *such that* $p > \overline{c}_i$ *and* $D_i(p) > S_\infty(p^-)$.

(iii) The highest i *satisfying the property in (ii) is equal to a constant* $\hat{\imath}_\infty$ *for all* p *in an open left-neighborhood* $\mathcal{V}$ *of* $\hat{p}_\infty$ *and*

$$\text{For each } p \in \mathcal{V}, \quad D_{\hat{\imath}_\infty}(p) - S_\infty(p^-) \in \arg\max\{B_\infty(p,s) : s \geq 0\}. \tag{24}$$

Using Lemma 5 along with the definition of B_∞ and the left-continuity of the mapping $p \mapsto S_\infty(p^-)$, we can select p_0 arbitrarily close to $\hat{p}_\infty$ such that: (1) $B^*_\infty(p_0) > 0$; (2) $p_0 \in \mathcal{V}$; (3) p_0 is a continuity point of S_∞; (4) p_0 is a multiple of Δ_n for n high enough. Any seller k can then deviate in Γ_n when price p_0 is quoted by supplying

$$\hat{s}^k_n \equiv [D_{\hat{\imath}_\infty}(p_0) - S_n(p_0^-) - s_n^{-k}(p_0)]^+$$

at price p_0 and nothing afterwards. Because $p_0 \in \mathcal{V}$, we have $D_{\hat{\imath}_\infty}(p_0) > S_\infty(p_0^-)$. Moreover, because p_0 is a continuity point of S_∞, we have $S_\infty(p_0^-) = S_\infty(p_0) = \lim_{n\to\infty} S_n(p_0)$. Finally, for each n, $S_n(p_0) \geq S_n(p_0^-) + s_n^{-k}(p_0)$ by definition. Thus, for n high enough, $\hat{s}^k_n$ is strictly positive, and this deviation is nontrivial.

How do the different types react to this deviation, and what is the impact on seller k's continuation profit at price p_0? Observe first that trading with any type $i \leq \hat{\imath}_\infty$ at price p_0 is always profitable as $c_i \leq \overline{c}_{\hat{\imath}_\infty} < p_0$. Thus, from seller k's perspective, any such type will at worst first exhaust his competitors' supply $s_n^{-k}(p_0)$ at price p_0 before purchasing anything from him. That is, her residual demand for the quantity $\hat{s}^k_n$ supplied by seller k at price p_0 is $[D_i(p_0) - S_n(p_0^-) - s_n^{-k}(p_0)]^+$. In particular, type $\hat{\imath}_\infty$ has a unique best response at price p_0 that involves purchasing $\hat{s}^k_n$ from seller k. By strict single-crossing, this a fortiori holds for types $i > \hat{\imath}_\infty$. Therefore, we can conclude that every seller k's continuation profit $\gamma^k_n(p_0)$ at price p_0 is at least $A_n(s_n^{-k}(p_0))$, where

$$\text{For each } s, \quad A_n(s) \equiv \sum_i m_i(p_0 - c_i)[\min\{D_i(p_0), D_{\hat{\imath}_\infty}(p_0)\} - S_n(p_0^-) - s]^+.$$

We now aggregate these profits. Because $A_n(s)$ is convex in s, we have

$$\sum_k A_n(s_n^{-k}(p_0)) \geq K A_n\left(\frac{1}{K}\sum_k s_n^{-k}(p_0)\right) = K A_n\left(\frac{K-1}{K}\, s_n(p_0)\right)$$

by Jensen's inequality. As p_0 is a continuity point of S_∞, $\lim_{n\to\infty} S_n(p_0^-) = S_\infty(p_0) = S_\infty(p_0^-)$ and $\lim_{n\to\infty} s_n(p_0) = 0$. Thus

$$\begin{aligned}
\lim_{n\to\infty} A_n\left(\frac{K-1}{K}\, s_n(p_0)\right) &= \sum_i m_i(p_0 - c_i)[\min\{D_i(p_0), D_{\hat{\imath}_\infty}(p_0)\} - S_\infty(p_0^-)]^+ \\
&= \sum_i m_i(p_0 - c_i)\min\{[D_i(p_0) - S_\infty(p_0^-)]^+, D_{\hat{\imath}_\infty}(p_0) - S_\infty(p_0^-)\} \\
&= B_\infty(p_0, D_{\hat{\imath}_\infty}(p_0) - S_\infty(p_0^-)) \\
&= B_\infty^*(p_0),
\end{aligned}$$

where the fourth equality follows from (24). Hence the aggregate equilibrium continuation profits at p_0 satisfy

$$\liminf_{n\to\infty} \sum_k \gamma_n^k(p_0) \geq K B_\infty^*(p_0) > 0. \tag{25}$$

Our goal in the remainder of the proof consists in deriving an upper bound on aggregate continuation profits that contradicts (25) for an appropriate choice of p_0.

To this end, we first provide an alternative expression for those profits. For each n, summing over the multiples $p \geq p_0$ of Δ_n and taking advantage of Lemma 3 and (23) yields

$$\begin{aligned}
\sum_k \gamma_n^k(p_0) &= \sum_{p\geq p_0}\sum_k \pi_n^k(p) \\
&= \sum_{\overline{p}\geq p\geq p_0} B_n(p, s_n(p)) \\
&= \sum_{\overline{p}\geq p\geq p_0}\sum_i m_i(p-c_i)\min\{[D_i(p) - S_n(p^-)]^+, S_n(p) - S_n(p^-)\} \qquad (26) \\
&= \sum_{\overline{p}\geq p\geq p_0}\sum_i m_i(p-c_i)[\min\{D_i(p), S_n(p)\} - S_n(p^-)]^+ \\
&= \sum_i \int_{[p_0,\overline{p}]} m_i(p-c_i)\, \boldsymbol{\sigma}_{i,n}(\mathrm{d}p),
\end{aligned}$$

where $\boldsymbol{\sigma}_{i,n}$ is the measure with finite support defined by

$$\text{For all } p \in [0,\overline{p}], \quad \boldsymbol{\sigma}_{i,n}(\{p\}) = [\min\{D_i(p), S_n(p)\} - S_n(p^-)]^+. \tag{27}$$

This is a Borel measure over $[0,\overline{p}]$ of at most mass $\overline{Q}$. As is customary, let us endow the space of such measures with the weak* topology generated by all continuous real-valued functions. The following lemma then characterizes the weak* limit of the sequence $(\boldsymbol{\sigma}_{i,n})_{n\in\mathbb{N}}$.

Lemma 6 *Let* $\overline{p}_{i,\infty} \equiv \inf\{p \in [0,\overline{p}] : S_\infty(p) \geq D_i(p)\}$. *Then the unique measure* $\boldsymbol{\sigma}_{i,\infty}$ *over the Borel sets of* $[0,\overline{p}]$ *such that*

$$\text{For each } p \in [0,\overline{p}], \quad \boldsymbol{\sigma}_{i,\infty}([0,p]) = \min\{S_\infty(p), D_i(\overline{p}_{i,\infty})\} \tag{28}$$

is the weak limit of the sequence* $(\boldsymbol{\sigma}_{i,n})_{n\in\mathbb{N}}$.

Because p_0 is a continuity point of S_∞ and hence not an atom of $\boldsymbol{\sigma}_{i,\infty}$ for all i, it follows from (26) and Lemma 6 that

$$\lim_{n\to\infty} \sum_k \gamma_n^k(p_0) = \sum_i \int_{[p_0,\overline{p}]} m_i(p - c_i)\, \boldsymbol{\sigma}_{i,\infty}(\mathrm{d}p).$$

The idea is now to cut this integral into two pieces. The following lemma reflects the intuitive idea that there are no profits to be earned at prices $p > \hat{p}_\infty$ as $B^*_\infty(p) = 0$ for any such p.

Lemma 7 *If* $p_1 > \hat{p}_\infty$, *then*

$$\sum_i \int_{(p_1,\overline{p}]} m_i(p - c_i)\, \boldsymbol{\sigma}_{i,\infty}(\mathrm{d}p) \leq 0.$$

Fix some $p_1 > \hat{p}_\infty$. Lemma 7 together with (25) implies

$$\begin{aligned} KB^*_\infty(p_0) &\leq \sum_i \int_{[p_0,p_1]} m_i(p - c_i)\, \boldsymbol{\sigma}_{i,\infty}(\mathrm{d}p) \\ &\leq \sum_i m_i(p_1 - c_i)\, \boldsymbol{\sigma}_{i,\infty}([p_0, p_1]) \\ &= \sum_i m_i(p_1 - c_i) \min\{[D_i(\overline{p}_{i,\infty}) - S_\infty(p_0)]^+, S_\infty(p_1) - S_\infty(p_0)\}, \end{aligned}$$

where the equality follows from (28) along with the continuity of S_∞ at p_0. Because $p_1 > \hat{p}_\infty$ is arbitrary and S_∞ is right-continuous, it follows that

$$\sum_i m_i(\hat{p}_\infty - c_i) \min\{[D_i(\overline{p}_{i,\infty}) - S_\infty(p_0)]^+, S_\infty(\hat{p}_\infty) - S_\infty(p_0)\} \geq KB^*_\infty(p_0),$$

where p_0 can be arbitrarily close to $\hat{p}_\infty$. We now prove that this inequality leads to a contradiction, which completes the proof of Theorem 4. Observe that, because $B_\infty(p, s)$ is left-continuous in p, we have $\lim_{p_0 \uparrow \hat{p}_\infty} B^*_\infty(p_0) = B^*_\infty(\hat{p}_\infty)$. We distinguish two cases.

Case 1 Suppose first that $B^*_\infty(\hat{p}_\infty) > 0$. Letting p_0 converge to $\hat{p}_\infty$ from below, we have

$$\sum_i m_i(\hat{p}_\infty - c_i) \min\{[D_i(\overline{p}_{i,\infty}) - S_\infty(\hat{p}^-_\infty)]^+, S_\infty(\hat{p}_\infty) - S_\infty(\hat{p}^-_\infty)\} \geq KB^*_\infty(\hat{p}_\infty). \tag{29}$$

If $\overline{p}_{i,\infty} = \hat{p}_\infty$, then it is obvious that each minimum in the left-hand side equals

$$\min\{[D_i(\hat{p}_\infty) - S_\infty(\hat{p}_\infty^-)]^+, S_\infty(\hat{p}_\infty) - S_\infty(\hat{p}_\infty^-)\}.$$

The same equality actually holds in all other cases. Indeed, if $\overline{p}_{i,\infty} < \hat{p}_\infty$, then $S_\infty(\hat{p}_\infty^-) \geq S_\infty(\overline{p}_{i,\infty}) \geq D_i(\overline{p}_{i,\infty}) > D_i(\hat{p}_\infty)$, and both minima are equal to zero, while, if $\overline{p}_{i,\infty} > \hat{p}_\infty$, then $D_i(\hat{p}_\infty) > D_i(\overline{p}_{i,\infty}) \geq S_\infty(\overline{p}_{i,\infty}^-) \geq S_\infty(\hat{p}_\infty) \geq S_\infty(\hat{p}_\infty^-)$, and both minima are equal to $S_\infty(\hat{p}_\infty) - S_\infty(\hat{p}_\infty^-)$. It follows that the left-hand side of (29) equals

$$\begin{aligned}
&\sum_i m_i(\hat{p}_\infty - c_i)\min\{[D_i(\hat{p}_\infty) - S_\infty(\hat{p}_\infty^-)]^+, S_\infty(\hat{p}_\infty) - S_\infty(\hat{p}_\infty^-)\} \\
&= B_\infty(\hat{p}_\infty, S_\infty(\hat{p}_\infty) - S_\infty(\hat{p}_\infty^-)) \\
&\leq B_\infty^*(\hat{p}_\infty),
\end{aligned}$$

which contradicts (29) as $K \geq 2$. This case is thus impossible.

Case 2 Suppose next that $B_\infty^*(\hat{p}_\infty) = 0$. Then, proceeding as in Lemma 5, we obtain $D_{\hat{\imath}_\infty}(\hat{p}_\infty) = S_\infty(\hat{p}_\infty^-)$. We have, for p_0 arbitrarily close to $\hat{p}_\infty$,

$$R(p_0) \equiv \frac{\sum_i m_i(\hat{p}_\infty - c_i)\min\{[D_i(\overline{p}_{i,\infty}) - S_\infty(p_0)]^+, S_\infty(\hat{p}_\infty) - S_\infty(p_0)\}}{\sum_i m_i(p_0 - c_i)\min\{[D_i(p_0) - S_\infty(p_0)]^+, D_{\hat{\imath}_\infty}(p_0) - S_\infty(p_0)\}} \geq K, \tag{30}$$

using (24) along with the continuity of S_∞ at p_0. It follows from $D_{\hat{\imath}_\infty}(\hat{p}_\infty) = S_\infty(\hat{p}_\infty^-)$ that $\overline{p}_{\hat{\imath}_\infty,\infty} = \hat{p}_\infty$ and $\overline{p}_{i,\infty} < \hat{p}_\infty$ for all $i < \hat{\imath}_\infty$. Hence, for any such i, $D_i(p_0) \leq D_i(\overline{p}_{i,\infty}) < S_\infty(p_0)$ for p_0 close enough to $\hat{p}_\infty$. Thus, for any such p_0, the denominator of $R(p_0)$ is equal to

$$\left(\sum_{i \geq \hat{\imath}_\infty} m_i\right)(p_0 - \overline{c}_{\hat{\imath}_\infty})[D_{\hat{\imath}_\infty}(p_0) - S_\infty(p_0)], \tag{31}$$

while the numerator of $R(p_0)$ is equal to

$$\sum_{i \geq \hat{\imath}_\infty} m_i(\hat{p}_\infty - c_i)\min\{[D_i(\overline{p}_{i,\infty}) - S_\infty(p_0)]^+, S_\infty(\hat{p}_\infty) - S_\infty(p_0)\}.$$

Because $\overline{c}_i \geq \hat{p}_\infty$ for all $i > \hat{\imath}_\infty$ by definition of $\hat{\imath}_\infty$ and $D_{\hat{\imath}_\infty}(\hat{p}_\infty) = D_{\hat{\imath}_\infty}(\overline{p}_{\hat{\imath}_\infty,\infty}) \leq S_\infty(\overline{p}_{\hat{\imath}_\infty,\infty}) = S_\infty(\hat{p}_\infty)$, the numerator of $R(p_0)$ is bounded above by

$$\left(\sum_{i \geq \hat{\imath}_\infty} m_i\right)(\hat{p}_\infty - \overline{c}_{\hat{\imath}_\infty})[D_{\hat{\imath}_\infty}(\hat{p}_\infty) - S_\infty(p_0)]. \tag{32}$$

Combining (30)–(32), we obtain

$$\frac{(\hat{p}_\infty - \overline{c}_{\hat{\imath}_\infty})[D_{\hat{\imath}_\infty}(\hat{p}_\infty) - S_\infty(p_0)]}{(p_0 - \overline{c}_{\hat{\imath}_\infty})[D_{\hat{\imath}_\infty}(p_0) - S_\infty(p_0)]} \geq R(p_0) \geq K,$$

a contradiction as $K \geq 2$ and p_0 can be arbitrarily close to $\hat{p}_\infty$. Hence the result. ∎

Supplementary Material (For Online Publication)

Appendix A: Proofs of Technical Results

Proof of Lemma 1. For the sake of clarity, the index i is hereafter omitted. The proof consists of three steps.

Step 1 We begin by analyzing the statement $p < \tau^T(q',t')$. By definition, $\tau^T(q',t')$ is the supremum of the set of prices p such that

$$\begin{aligned}\max\{u(q+q',T(q)+t'):q\} &= u^T(q',t')\\ &< \max\{u^T(q'+q'',t'+pq''):q''\}\\ &= \max\{u(q+q'+q'',T(q)+t'+pq''):q,q''\}\\ &= \max\{u(q+q',T\,\square\,T_p(q)+t'):q\},\end{aligned}$$

where T_p is the linear tariff with slope p and $T\,\square\,T_p(q) \equiv \min\{T(q')+p(q-q'):q'\in[0,q]\}$ is the infimal convolution of T and T_p (Rockafellar (1970, Theorem 5.4)). Thus the statement $p < \tau^T(q',t')$ is equivalent to

$$\max\{u(q+q',T(q)+t'):q\} < \max\{u(q+q',T\,\square\,T_p(q)+t'):q\}. \tag{A.1}$$

Notice that $T\,\square\,T_p \leq T$, and that both tariffs coincide up to some quantity q_p, beyond which the inequality is strict. Two cases can arise. Either the maximization problem on the right-hand side of (A.1) admits one solution at most equal to q_p. Then (A.1) is an equality. Or all the solutions to this problem are strictly higher than q_p. Then (A.1) cannot be an equality because, if it were, then there would exist a solution to the maximization problem on the left-hand side of (A.1) at most equal to q_p, and thus this solution would also be a solution to the maximization problem on the right-hand side of (A.1), a contradiction; therefore, (A.1) must hold because in any case $T\,\square\,T_p \leq T$. Overall, we have shown that the statement $p < \tau^T(q',t')$ is equivalent to the statement that all the solutions to the maximization problem on the right-hand side of (A.1) are strictly higher than q_p.

Step 2 Next, fix t' and, for any quantities q'_0 and q'_1 such that $q'_0 < q'_1$, define the following quasiconcave functions:

$$v_0(q,t) \equiv u(q+q'_0,t+t') \text{ and } v_1(q,t) \equiv u(q+q'_1,t+t').$$

Assumption 2 expresses that the indifference curves for v_0 are everywhere steeper than the indifference curves for v_1. Therefore, if two buyers with utilities v_0 and v_1 face the same tariff $t = T(q)$, then the lowest optimal quantity choice for the buyer with utility v_0 is at least as large as the lowest optimal quantity choice for the buyer with utility v_1.

Step 3 Now, suppose that $p < \tau^T(q'_1, t')$. From Step 1, we first obtain that all the solutions to the maximization problem on the right-hand side of (A.1) (with q' replaced by q'_1) are strictly higher than q_p. From Step 2, we next obtain that all the solutions to the maximization problem on the right-hand side of (A.1) (with q' replaced by q'_0) are strictly higher than q_p. From Step 1 again, we finally obtain $p < \tau^T(q'_0, t')$. Because p is arbitrary, this shows that $\tau^T(q'_1, t') \leq \tau^T(q'_0, t')$ for all t' and $q'_0 < q'_1$, and, therefore, that the property expressed by Assumption 2 is inherited by $\tau^T(q', t')$ from $\tau(q, t)$; a fortiori, Assumption 1 holds for $\tau^T(q', 0)$. The result follows. ■

Proof of Lemma 3. By assumption, there exists a type i such that $D_i(\bar{c}_i) > 0$. Thus, as D_i is continuous, there exists n_0 such that $D_i(\bar{c}_i + \Delta_{n_0}) > 0$; define $p \equiv \bar{c}_i + \Delta_{n_0}$. Because $\bar{c}_i$ belongs to the price grid with tick size Δ and the price grids for tick sizes $\Delta_n = \Delta/2^n$ are nested, p belongs to the price grid with tick size Δ_n for all $n \geq n_0$.

Fix some $n \geq n_0$, and an equilibrium of Γ_n. Suppose first that the aggregate quantity Q purchased by type I in equilibrium satisfies $Q < D_i(p)$. Then $Q < D_I(p)$ by single-crossing. As type I overall purchases Q, the aggregate supply at prices lower than or equal to p must be such that $S(p) \leq Q$. In equilibrium, aggregate revenues are constrained by individual rationality, because no type would accept to pay more than $U_I(Q) - U_I(0)$. By ignoring costs, we obtain that aggregate expected profits are at most $U_I(Q) - U_I(0)$.

Now, any seller can deviate when price $p > \bar{c}_1$ is quoted by supplying $D_i(p) - Q > 0$ at price p and nothing afterwards. The aggregate supply at prices $p' < p$ is unchanged, and is at most $S(p) \leq Q$. Thus, as revenues are nonnegative, the deviating seller's expected profit at prices $p' < p$ is at worst $-\bar{c}_I Q$. Second, trading with any type $j < i$ at price p is always profitable as $c_j \leq \bar{c}_i < p$. Thus, from the deviating seller's perspective, these types will at worst choose not to trade with him at price p. Third, the aggregate supply at prices $p' \leq p$ following the deviation is at most $S(p) + D_i(p) - Q \leq D_i(p)$. Thus type i has a unique best response at price p that involves purchasing $D_i(p) - Q$ from the deviating seller. By strict single-crossing, this a fortiori holds for types $j > i$. Finally, the deviating seller earns zero profits at prices $p' > p$. In equilibrium, this deviation cannot be profitable, so that a fortiori

$$-\bar{c}_I Q + \left(\sum_{j \geq i} m_j\right)(p - \bar{c}_i)[D_i(p) - Q] \leq U_I(Q) - U_I(0).$$

Because $D_i(p) > 0$ and $p > \bar{c}_i$, this inequality is violated at $Q = 0$. This shows that the aggregate quantity Q purchased by type I is bounded away from zero in all equilibria in which $Q < D_i(p)$. As $Q \geq D_i(p) > 0$ in all other equilibria, it follows that there exists $\underline{Q} > 0$ such that type I purchases at least $\underline{Q}$ in any equilibrium. In particular, because $D_i(\overline{c}_i) > 0$ by assumption, we can select $\underline{Q} < D_i(\overline{c}_i)$; and we can select $\underline{Q}$ independently of the equilibrium of Γ_n, and independently of $n \geq n_0$.

Finally, because type I purchases at least $\underline{Q}$, she is not willing to trade at prices $p > \overline{p} \equiv U_I'(\underline{Q})$. Moreover, as expected profits are nonnegative in equilibrium, she must purchase her aggregate quantity Q at a price at least equal to $\overline{c}_1$. This implies that she purchases at most $\overline{Q} \equiv D_I(\overline{c}_1)$, which is finite and strictly higher than $\underline{Q}$. The result follows. ■

Proof of Lemma 4. Suppose first that (22) holds and that the tariff T implements a budget-feasible allocation $(q_i, T(q_i))_{i=1}^I$. Then, for all i and p, $p > \overline{c}_i$ implies $D_i(p) \leq S(p^-)$; otherwise, $B(p, s)$ would be strictly positive for s small enough, a contradiction. Thus no type i is willing to trade at prices $p > \overline{c}_i$ along T; that is, for each i, $U_i'(q_i) \leq \overline{c}_i$ and $T(q_i) - T(q_{i-1}) \leq \overline{c}_i(q_i - q_{i-1})$. By a now standard argument, budget-feasibility implies that these last inequalities hold as equalities. If $q_{i-1} = q_i$, then we obtain $U_i'(q_{i-1}) \leq \overline{c}_i$, and (ii) in Theorem 2 holds. If $q_{i-1} < q_i$, then, because $\partial^- T(q_i) \leq U_i'(q_i) \leq \overline{c}_i$ and T is convex with $T(q_i) - T(q_{i-1}) = \overline{c}_i(q_i - q_{i-1})$, it must be that T is affine with slope $\overline{c}_i$ over the interval $[q_{i-1}, q_i]$, as required by (iii) in Theorem 2; hence $\partial^- T(q_i) = U_i'(q_i) = \overline{c}_i$, and (ii) in Theorem 2 again holds. Thus $(q_i, T(q_i))_{i=1}^I$ is the JHG allocation, and T is, up to inessential modifications beyond q_I, the JHG tariff. Conversely, consider the JHG tariff T and the JHG allocation $(Q_i, T_i)_{i=1}^I$ it implements, which is budget-balanced by construction. From (ii) in Theorem 2, $U_i'(Q_i) \leq \overline{c}_i$ and $Q_i \leq S(\overline{c}_i)$ for all i. Consider any price p. If $p \leq \overline{c}_1$, then (22) clearly holds. If $p > \overline{c}_i$ for some i, then $p > U_i'(Q_i)$ and hence $D_i(p) \leq Q_i \leq S(\overline{c}_i) \leq S(p^-)$, so that (22) again holds. The result follows. ■

Proof of Lemma 5. We prove each statement in turn.

(i) By Lemma 3, we have $D_I(p) \leq S_n(p^-)$ for all $p > \overline{p}$ and n large enough. Taking limits, this implies that $D_I(p) \leq S_\infty(p^-)$ at any continuity point $p > \overline{p}$ of S_∞, so that $D_i(p) \leq D_I(p) \leq S_\infty(p^-)$ for all i and $p > \overline{p}$ by continuity and single-crossing. Therefore, $B_\infty^*(p) = 0$ for all $p > \overline{p}$ by definition of B_∞, and hence $\hat{p}_\infty \leq \overline{p}$ by definition of $\hat{p}_\infty$.

(ii) Summing by parts the expression (22) for $B_\infty(p, s)$ yields

$$\sum_i \left(\sum_{j \geq i} m_j \right) (p - \overline{c}_i)(\min\{[D_i(p) - S_\infty(p^-)]^+, s\} - \min\{[D_{i-1}(p) - S_\infty(p^-)]^+, s\}),$$

which is at most zero if $p > \overline{c}_i$ implies $D_i(p) \leq S_\infty(p^-)$ for all i.

(iii) The set $\mathcal{I}_\infty \equiv \{i : \hat{p}_\infty > \overline{c}_i \text{ and } D_i(\hat{p}_\infty) \geq S_\infty(\hat{p}_\infty^-)\}$ must be nonempty; otherwise, for all i and $p < \hat{p}_\infty$ close enough to $\hat{p}_\infty$, $p > \overline{c}_i$ would imply $D_i(p) < S_\infty(p^-)$, contradicting (ii) and the definition of $\hat{p}_\infty$. Define $\hat{\imath}_\infty \equiv \max \mathcal{I}_\infty$. Then $\overline{c}_{\hat{\imath}_\infty} < \hat{p}_\infty \leq \overline{c}_{\hat{\imath}_\infty+1}$ and $D_{\hat{\imath}_\infty}(\hat{p}_\infty) \geq S_\infty(\hat{p}_\infty^-)$, so that, for each $p < \hat{p}_\infty$ close enough to $\hat{p}_\infty$, $\overline{c}_{\hat{\imath}_\infty} < p \leq \overline{c}_{\hat{\imath}_\infty+1}$ and $D_{\hat{\imath}_\infty}(p) > S_\infty(p^-)$. Hence, for any such p, $\hat{\imath}_\infty$ is the highest i satisfying the property in (ii). This holds for p in an open left-neighborhood of $\hat{p}_\infty$, because, if this holds for some p, then $\overline{c}_{\hat{\imath}_\infty} < p' < p \leq \overline{c}_{\hat{\imath}_\infty+1}$ and $D_{\hat{\imath}_\infty}(p') - S_\infty(p'^-) > D_{\hat{\imath}_\infty}(p) - S_\infty(p^-) > 0$ for all p' close enough to p. Finally, (24) is a direct consequence of (22) and of the definition of $\hat{\imath}_\infty$. The result follows. ■

Proof of Lemma 6. Observe first that, by Lemma 3, $D_I(p) \leq S_n(p)$ for all $p > \overline{p}$ and n large enough. As in the proof of Lemma 5(i), this implies that $D_i(\overline{p}) \leq S_n(\overline{p})$ and $D_i(\overline{p}) \leq S_\infty(\overline{p})$ for all i by continuity and single-crossing. Thus $\overline{p}_{i,\infty}$ is well defined. That condition (28) determines a unique measure over the Borel sets of $[0, \overline{p}]$ is standard (Billingsley (1995, Theorem 12.4)). We must show that $(\boldsymbol{\sigma}_{i,n}([0,p]))_{n\in\mathbb{N}}$ converges to $\boldsymbol{\sigma}_{i,\infty}([0,p])$ at any continuity point p of $p \mapsto \boldsymbol{\sigma}_{i,\infty}([0,p])$. By (28), the set of such points is included in the set of continuity points of S_∞. Moreover, using the definition (27) of $\boldsymbol{\sigma}_{i,n}$, we can check that

$$\text{For each } p \in [0, \overline{p}], \quad \boldsymbol{\sigma}_{i,n}([0,p]) = \min\{S_n(p), D_i(\overline{p}_{i,n})\},$$

where, according to our preliminary observation, $\overline{p}_{i,n} \equiv \inf\{p \in [0, \overline{p}] : S_n(p) \geq D_i(p)\}$ is well defined for n large enough. As D_i is continuous and $\lim_{n\to\infty} S_n(p) = S_\infty(p)$ at any continuity point of S_∞, we thus only need to prove that $\lim_{n\to\infty} \overline{p}_{i,n} = \overline{p}_{i,\infty}$. Hence consider a subsequence of $(\overline{p}_{i,n})_{n\in\mathbb{N}}$ whose elements all satisfy $\overline{p}_{i,n} < \overline{p}_{i,\infty}$, and suppose, by way of contradiction, that it does not converge to $\overline{p}_{i,\infty}$. Thus there exists $\varepsilon > 0$ and a subsubsequence whose elements all satisfy $\overline{p}_{i,n} < \overline{p}_{i,\infty} - \varepsilon$. Using the definition of $\overline{p}_{i,n}$ and $\overline{p}_{i,\infty}$ and the monotonicity of supply and demand functions, we then obtain that for each $p \in (\overline{p}_{i,\infty} - \varepsilon, \overline{p}_{i,\infty})$, we have $S_n(p) \geq D_i(p) > S_\infty(p)$ for any such n, a contradiction as $\lim_{n\to\infty} S_n(p) = S_\infty(p)$ if p is a continuity point of S_∞. A symmetric argument applies to a subsequence of $(\overline{p}_{i,n})_{n\in\mathbb{N}}$ whose elements all satisfy $\overline{p}_{i,n} > \overline{p}_{i,\infty}$. The result follows. ■

Proof of Lemma 7. Because $\overline{p}_{j,\infty}$ is nondecreasing in j, we can partition the integration interval into (i) successive intervals $(\overline{p}_{j-1,\infty}, \overline{p}_{j,\infty})$, on which $\boldsymbol{\sigma}_{i,\infty}$ puts a mass if and only if $i \geq j$ by (28), and (ii) possible mass points at each bound, once more using (28) to compute the mass. To avoid double-counting in (ii), we let A be the set of $\overline{p}_{j,\infty}$, and for $p \in A$ we let $j(p)$ be the lowest type such that $\overline{p}_{j,\infty} = p$. We obtain

$$\sum_i \int_{(p_1, \overline{p}]} m_i(p - c_i)\, \boldsymbol{\sigma}_{i,\infty}(\mathrm{d}p)$$

$$= \sum_j \int_{(\max\{\overline{p}_{j-1,\infty}, p_1\}, \max\{\overline{p}_{j,\infty}, p_1\})} \left[\sum_{i \geq j} m_i(p - c_i)\right] \mathrm{d}S_\infty(p)$$
$$+ \sum_{p \in A, p > p_1} \sum_{j \geq j(p)} m_j(p - c_j)[\min\{D_j(p), S_\infty(p)\} - S_\infty(p^-)]^+,$$

with $\int_\emptyset \equiv 0$. For each integral on the right-hand side, when $p < \overline{p}_{j,\infty}$ we have $D_j(p) > S_\infty(p) \geq S_\infty(p^-)$; but $p > p_1 > \hat{p}_\infty$ implies $B^*_\infty(p) = 0$, so that $D_j(p) > S_\infty(p^-)$ implies $p \leq \overline{c}_j$ by Lemma 5(ii). Each of these integrals is thus at most zero. For the second term on the right-hand side, fix $p \in A$, $p > p_1$. Then we have $p = \overline{p}_{j(p),\infty}$, and for $j < j(p)$ we have $p > \overline{p}_{j,\infty}$ by definition of $j(p)$. Therefore, $\boldsymbol{\sigma}_{j,\infty}$ puts no mass on p if $j < j(p)$, so that we can extend the sum $\sum_{j \geq j(p)}$ to all types. Hence this sum is equal to

$$\sum_j m_j(p - c_j) \min\{[D_j(p) - S_\infty(p^-)]^+, [S_\infty(p) - S_\infty(p^-)]^+\} = B_\infty(p, S_\infty(p) - S_\infty(p^-)),$$

which is at most zero as $p_1 > \hat{p}_\infty$. The result follows. ■

Appendix B: Arbitrary Distributions

In this appendix, we extend Theorem 1 to arbitrary distributions of types with bounded support $\mathcal{I}$ over the real line. Denote by i the buyer's type, and by $\mathbf{m}$ the corresponding distribution; $\mathbf{m}$ may be continuous, discrete, or mixed. It will sometimes be convenient to think of any point in $\mathcal{I}_0 \equiv [\min \mathcal{I}, \max \mathcal{I}]$ as a type, even if it does not belong to $\mathcal{I}$. We impose the same conditions on the utility functions u_i and on the upper-tail conditional expectations of unit costs $\overline{c}_i^{\mathbf{m}} \equiv \mathbf{E}^{\mathbf{m}}[c_j \,|\, j \geq i]$ as in Section 2, and we moreover assume that $u_i(q, t)$ is jointly continuous in (i, q, t) and that c_i is continuous in i.

The proof that Condition EP is necessary for entry-proofness is exactly the same as in Section 3. There only remains to show that Condition EP is sufficient for entry-proofness. According to the taxation principle, there is no loss of generality in letting the entrant offer a tariff specifying a transfer $T(q)$ to be paid as a function of the quantity q demanded by the buyer, with $T(0) \equiv 0$. We assume that the domain of T is a compact set containing 0 and that T is bounded from below and lower semicontinuous. These minimal regularity conditions ensure that any type i's maximization problem

$$\max\{u_i(q, T(q)) : q \geq 0\} \tag{B.1}$$

has a solution. The following result then holds.

Lemma B.1 *There exists for each i a solution q_i to (B.1) such that* *(i) The mapping $i \mapsto q_i$ is nondecreasing.*

(ii) The mapping $i \mapsto T(q_i) - c_i q_i$ is bounded from below and lower semicontinuous.

Proof. As in Step 1 of the proof of Theorem 1, the weak single-crossing condition ensures that we can select the buyer's best response in such a way that the mapping $i \mapsto q_i$ is nondecreasing. This implies (i). As for (ii), observe first that, because T has a compact domain and is bounded from below, the mapping $i \mapsto T(q_i) - c_i q_i$ is bounded from below no matter the buyer's best response. To show that the buyer's best response can be chosen in such a way that this mapping is lower semicontinuous, it is useful to fix a best response $i \mapsto q_i$ and some type $i_0 \in \mathcal{I}_0$, and then to distinguish two cases.

Case 1 Suppose first that $i \mapsto q_i$ is continuous at i_0. Then, as T is lower semicontinuous and c_i is continuous in i, we have $\liminf_{i \to i_0} T(q_i) - c_i q_i \geq T(q_{i_0}) - c_{i_0} q_{i_0}$.

Case 2 Suppose next that $i \mapsto q_i$ is discontinuous and left-continuous at i_0. (The other types of jump discontinuities can be treated in a similar way.) Because the domain of T is a compact set, it must include $q_{i_0}^+ \equiv \lim_{i \downarrow i_0} q_i$; moreover, T must be right-continuous at $q_{i_0}^+$; otherwise, some type $i > i_0$ would be strictly better off purchasing $q_{i_0}^+$ instead of q_i, a contradiction. Now, type i_0 must be indifferent between the trades $(q_{i_0}, T(q_{i_0}))$ and $(q_{i_0}^+, T(q_{i_0}^+))$. Indeed, we clearly have $u_{i_0}(q_{i_0}, T(q_{i_0})) \geq u_{i_0}(q_{i_0}^+, T(q_{i_0}^+))$ and, if we had $u_{i_0}(q_{i_0}, T(q_{i_0})) > u_{i_0}(q_{i_0}^+, T(q_{i_0}^+))$, then, by continuity of u_i in i, some type $i > i_0$ would be strictly better off purchasing q_{i_0} instead of q_i, a contradiction. We can thus select the trade of type i_0 so that $\liminf_{i \to i_0} T(q_i) - c_i q_i \geq T(q_{i_0}) - c_{i_0} q_{i_0}$. The result follows. ■

The next step of the analysis consists in checking that any distribution that satisfies Condition EP can be weakly approximated by a sequence of discrete distributions that satisfy Condition EP. Specifically, the following result holds.

Lemma B.2 *If $\mathbf{m}$ satisfies Condition EP, then there exists a sequence $(\mathbf{m}_n)_{n \in \mathbb{N}}$ of discrete distributions that weakly converges to $\mathbf{m}$ and such that*

$$\textit{For all } n \textit{ and } i,\ \overline{c}_i^{\mathbf{m}_n} \geq \overline{c}_i^{\mathbf{m}}.$$

Proof. The proof is a simple adaptation of Hendren (2013, Supplementary Material, Lemma A.7), using the fact that c_i is continuous in i and that, as $\overline{c}_i^{\mathbf{m}}$ is nondecreasing in i, $c_{\max \mathcal{I}} \geq c_i$ for all i. Hendren's (2013) proof establishes that the sequence of cumulative distribution functions associated to the sequence $(\mathbf{m}_n)_{n \in \mathbb{N}}$ can be chosen so as to uniformly converge to the cumulative distribution function associated to $\mathbf{m}$. The result follows. ■

We are now ready to complete the proof of Theorem 1 for arbitrary distributions. Let $\mathbf{m}$ be a distribution that satisfies Condition EP. Fix a tariff T as above and, for each i, a solution q_i to (B.1) such that properties (i)–(ii) in Lemma B.1 hold. Lemma B.2 implies that there exists a sequence of discrete distributions $(\mathbf{m}_n)_{n\in\mathbb{N}}$ that weakly converges to $\mathbf{m}$ and such that each $\mathbf{m}_n$ satisfies Condition EP. Taking advantage of the fact that the mapping $i \mapsto q_i$ is nondecreasing, we can apply the version of Theorem 1 for discrete distributions provided in the main text to obtain

$$\text{For each } n, \int [T(q_i) - c_i q_i]\, \mathbf{m}_n(\mathrm{d}i) \leq 0.$$

Because the mapping $i \mapsto T(q_i) - c_i q_i$ is bounded from below and lowersemicontinuous, the weak convergence of the sequence $(\mathbf{m}_n)_{n\in\mathbb{N}}$ to $\mathbf{m}$ then yields

$$\int [T(q_i) - c_i q_i]\, \mathbf{m}(\mathrm{d}i) \leq \liminf_{n\to\infty} \int [T(q_i) - c_i q_i]\, \mathbf{m}_n(\mathrm{d}i) \leq 0$$

according to a corollary of the portmanteau theorem (Aliprantis and Border (2006, Theorem 15.5)). Hence, if the distribution $\mathbf{m}$ satisfies Condition EP, no tariff can guarantee the entrant a strictly positive expected profit, which is the desired result.

Appendix C: Examples

The following examples for the buyer's preferences illustrate the range of possible applications of our model.

Quasilinear Utility We may first suppose, as in the models of trade on financial markets studied by Glosten (1989, 1994), Biais, Martimort, and Rochet (2000), Mailath and Nöldeke (2008), and Back and Baruch (2013), that every type i's preferences are quasilinear,

$$u_i(q,t) \equiv U_i(q) - t,$$

for some concave utility function U_i. The weak single-crossing condition is satisfied if $\partial^+ U_i(q)$ is nondecreasing in i for all q, and the concavity of U_i ensures that Assumption 1 holds.

We now consider variations on the standard Rothschild and Stiglitz (1976) insurance economy, in which the buyer has initial wealth w_0 and faces the risk of a loss l.

Expected Utility Given a loss probability c_i, every type i's preferences over coverage-premium pairs (q,t) have an expected-utility representation

$$u_i(q,t) \equiv c_i u(w_0 - l + q - t) + (1 - c_i) u(w_0 - t),$$

for some strictly increasing and strictly concave von Neumann–Morgenstern utility function u. The weak single-crossing condition is satisfied if c_i is nondecreasing in i, and the strict concavity of u ensures that Assumption 1 holds.

Rank-Dependent Expected Utility (Quiggin (1982)) Given a loss probability c_i, every type i's preferences over coverage-premium pairs (q,t) have a rank-dependent expected-utility representation

$$u_i(q,t) \equiv [w(1) - w(1-c_i)]u(w_0 - l + q - t) + w(1-c_i)u(w_0 - t),$$

for some strictly increasing and strictly concave von Neumann–Morgenstern utility function u and some strictly increasing weighting function w such that $w(0) \equiv 0$ and $w(1) \equiv 1$. Because w is strictly increasing, the weak single-crossing condition is satisfied if c_i is nondecreasing in i, and the strict concavity of u ensures that Assumption 1 holds.

Robust Control (Hansen and Sargent (2007)) Every type i now recognizes that the true probability distribution over outcomes $\tilde{\mathbf{c}}_i \equiv (\tilde{c}_i, 1-\tilde{c}_i)$ is uncertain and may differ from $\mathbf{c}_i \equiv (c_i, 1-c_i)$, and her preferences over coverage-premium pairs (q,t) have a robust-control representation

$$u_i(q,t) \equiv \min\left\{\tilde{c}_i u(w_0 - l + q - t) + (1-\tilde{c}_i)u(w_0 - t) + \alpha e(\tilde{\mathbf{c}}_i, \mathbf{c}_i) : \tilde{\mathbf{c}}_i\right\},$$

for some strictly increasing and strictly concave von Neumann–Morgenstern utility function u, where $e(\tilde{\mathbf{c}}_i, \mathbf{c}_i)$ is the relative entropy function that penalizes distortions from $\mathbf{c}_i$,

$$e(\tilde{\mathbf{c}}_i, \mathbf{c}_i) \equiv \tilde{c}_i \log_2\left(\frac{\tilde{c}_i}{c_i}\right) + (1-\tilde{c}_i)\log_2\left(\frac{1-\tilde{c}_i}{1-c_i}\right).$$

As u_i is a minimum of concave functions, it is itself concave. For each (q,t), let us denote by $\tilde{\mathbf{c}}_i(q,t) = (\tilde{c}_i(q,t), 1-\tilde{c}_i(q,t))$ the unique solution to the minimization problem that defines $u_i(q,t)$. Taking first-order conditions yields

$$\frac{1-\tilde{c}_i(q,t)}{\tilde{c}_i(q,t)} = \frac{1-c_i}{c_i}\, 2^{u(w_0-l+q-t)-u(w_0-t)}.$$

Thus, for each (q,t), $\tilde{c}_i(q,t)$ and c_i are comonotonic in i, and $\tilde{c}_i(q,t)$ is strictly decreasing in q for all t. Assuming that u is differentiable, the marginal rate of substitution of u_i is

$$\tau_i(q,t) = \left[1 + \frac{1-\tilde{c}_i(q,t)}{\tilde{c}_i(q,t)}\, \frac{u'(w_0-t)}{u'(w_0-l+q-t)}\right]^{-1}.$$

Hence, because $\tilde{c}_i(q,t)$ and c_i are comonotonic in i, the weak single-crossing condition is satisfied if c_i is nondecreasing in i. Finally, together with the strict concavity of u, the fact that $\tilde{c}_i(q,0)$ is strictly decreasing in q ensures that Assumption 1 holds.

Smooth Ambiguity Aversion (Klibanoff, Marinacci, and Mukerji (2005)) Every type i now believes that his true loss probability c has continuous density f_i over $[0,1]$, and her preferences over coverage-premium pairs (q,t) have a smooth-ambiguity-aversion representation

$$u_i(q,t) \equiv \int \phi(cu(w_0 - l + q - t) + (1-c)u(w_0 - t))f_i(c)\,\mathrm{d}c,$$

for some strictly increasing and strictly concave von Neumann–Morgenstern utility function u and some strictly increasing and strictly concave function ϕ capturing ambiguity aversion regarding c. As u and ϕ are concave, so is u_i. Assuming that u and ϕ are differentiable with bounded derivatives, the marginal rate of substitution of u_i is

$$\tau_i(q,t) = \left[1 + \frac{u'(w_0 - t)}{u'(w_0 - l + q - t)} \times \frac{\int \phi'(cu(w_0 - l + q - t) + (1-c)u(w_0 - t))(1-c)f_i(c)\,\mathrm{d}c}{\int \phi'(cu(w_0 - l + q - t) + (1-c)u(w_0 - t))cf_i(c)\,\mathrm{d}c}\right]^{-1}.$$

We claim that, if the densities f_i are nondecreasing in the monotone-likelihood-ratio order, then the weak single-crossing condition is satisfied. To see this, observe that

$$\frac{\int \phi'(cu(w_0 - l + q - t) + (1-c)u(w_0 - t))(1-c)f_i(c)\,\mathrm{d}c}{\int \phi'(cu(w_0 - l + q - t) + (1-c)u(w_0 - t))cf_i(c)\,\mathrm{d}c} = \frac{1}{\int c\,\mathrm{d}G_i(c)} - 1,$$

where G_i is a distribution with density

$$g_i(c) \equiv \frac{\phi'(cu(w_0 - l + q - t) + (1-c)u(w_0 - t))f_i(c)}{\int \phi'(cu(w_0 - l + q - t) + (1-c)u(w_0 - t))f_i(c)\,\mathrm{d}c}.$$

In particular, if the densities f_i are nondecreasing in the monotone-likelihood ratio order, so are the densities g_i. This implies that the ratio $1/\int c\,\mathrm{d}G_i(c)$ is nonincreasing in i, which proves the claim given the expression for $\tau_i(q,t)$. Finally, there remains to determine when Assumption 1 holds. Letting $t \equiv 0$ and proceeding as above yields

$$\frac{\int \phi'(cu(w_0 - l + q) + (1-c)u(w_0))(1-c)f_i(c)\,\mathrm{d}c}{\int \phi'(cu(w_0 - l + q) + (1-c)u(w_0))cf_i(c)\,\mathrm{d}c} = \frac{1}{\int c\,\mathrm{d}G_i(c\,|\,q)} - 1,$$

where, for each q, $G_i(\cdot\,|\,q)$ is a distribution with density

$$g_i(c\,|\,q) \equiv \frac{\phi'(cu(w_0 - l + q) + (1-c)u(w_0))f_i(c)}{\int \phi'(cu(w_0 - l + q) + (1-c)u(w_0))f_i(c)\,\mathrm{d}c}.$$

It follows that

$$\frac{g_i(c\,|\,0)}{g_i(c\,|\,q)} \propto \frac{\phi'(cu(w_0 - l) + (1-c)u(w_0))}{\phi'(cu(w_0 - l + q) + (1-c)u(w_0))},$$

up to multiplicative constants. In particular,

$$\frac{\partial}{\partial c}\left[\frac{g_i(c\,|\,0)}{g_i(c\,|\,q)}\right] \propto -\frac{\phi''(cu(w_0-l)+(1-c)u(w_0))}{\phi'(cu(w_0-l)+(1-c)u(w_0))}\,[u(w_0)-u(w_0-l)]$$
$$+\frac{\phi''(cu(w_0-l+q)+(1-c)u(w_0))}{\phi'(cu(w_0-l+q)+(1-c)u(w_0))}\,[u(w_0)-u(w_0-l+q)],$$

which is strictly positive for $q>0$ if the function ϕ features nonincreasing concavity in the sense that $-\phi''/\phi'$ is nonincreasing. Under these circumstances, $g_i(c\,|\,0)$ dominates $g_i(c\,|\,q)$ in the monotone-likelihood-ratio order and, as a result, $\int c\,\mathrm{d}G_i(c\,|\,q) < \int c\,\mathrm{d}G_i(c\,|\,0)$. Combining this with the straightforward observation that $u'(w_0-l+q) < u'(w_0-l)$ by strict concavity of u, we obtain, using the above expression for $\tau_i(q,t)$, that $\tau_i(q,0)<\tau_i(0,0)$ for all $q>0$, so that Assumption 1 holds.

It is easy to check that the slightly stronger Assumption 2 used in Section 4 holds in the above examples. We can also modify these examples to allow for multiple loss levels by focusing on coinsurance contracts requiring that a share q of the loss be covered for a premium t. For instance, in the expected-utility model, every type i now faces the risk of a loss l distributed according to a continuous density f_i over $[0,1]$, and her preferences are represented by

$$u_i(q,t) \equiv \int u(w_0-(1-q)l-t)f_i(l)\,\mathrm{d}l.$$

The weak single-crossing condition is satisfied if the densities f_i are nondecreasing in the monotone-likelihood ratio order (Attar, Mariotti, and Salanié (2019, Appendix B)). Finally, many other families of preferences, involving, for instance, first-order risk aversion (Segal and Spivak (1990)), also fit within our general framework.

Appendix D: Counterexamples

The following example justifies the claim made in Section 3 that, when Assumption 1 does not hold, entry with a menu of contracts can be profitable even though Condition EP is satisfied.

Example 1 Consider a two-type economy in which each type has preferences represented by $u_i(q,t) \equiv (q+1)(\theta_i q - t)$, where $\theta_2 > \theta_1 > 0$. These preferences are convex, with

$$\tau_i(q,t) = \theta_i\left(1+\frac{q}{q+1}\right) - \frac{t}{q+1}, \tag{D.1}$$

so that the strict single-crossing condition is satisfied. However, by (D.1), $\tau_i(q,0)$ is strictly increasing in q, so that Assumption 1 does not hold. Now, fix quantities $q_2 > q_1 > 0$ and, for some small $\eta > 0$, consider an entrant offering a menu $\{(q_1, t_1), (q_2, t_2)\}$ such that

$$t_1 \equiv \theta_1 q_1 - \eta,$$

so that type 1 earns a small rent compared to $u_1(0, 0) = 0$, and

$$t_2 \equiv \theta_2 q_2 - \frac{q_1 + 1}{q_2 + 1}\,(\theta_2 q_1 - t_1) - \eta,$$

so that type 2 has a slight preference for (q_2, t_2) over (q_1, t_1). Hence each type has a unique best response, and the entrant's expected profit is $m_1(t_1 - c_1 q_1) + m_2(t_2 - c_2 q_2)$. To compute this expected profit, set up costs so that $\bar{c}_1 \equiv \theta_1 + \varepsilon$ and $\bar{c}_2 \equiv \theta_2 + \varepsilon$ for some small $\varepsilon > 0$. Notice that, by (D.1) again, Condition EP is satisfied. As in (2), the entrant's expected profit can be rewritten as $t_1 - \bar{c}_1 q_1 + m_2[t_2 - t_1 - \bar{c}_2(q_2 - q_1)]$; this in turn simplifies into

$$m_2(\bar{c}_2 - \bar{c}_1)(q_2 - q_1)\,\frac{q_1}{q_2 + 1} - \varepsilon(m_1 q_1 + m_2 q_2) - \eta\left(1 + m_2\,\frac{q_1 + 1}{q_2 + 1}\right),$$

which is strictly positive for arbitrary quantities $q_2 > q_1 > 0$ whenever ε and η are small enough. This proves the claim. Notice that the entrant makes a profit when trading with type 1 and a loss when trading with type 2; but he also incurs an expected loss on the quantity layer q_1, which he more than recoups on the quantity layer $q_2 - q_1$.

For the study of market breakdown in Section 3.2, the first difficulty is that there may exist menus of contracts for which the buyer has multiple best responses, some of which may be more favorable to the entrant than others. This difficulty can be overcome by requiring that types be ordered according to the strict single-crossing condition. This assumption is tight. Indeed, the following example shows that, when types are only ordered according to the weak single-crossing condition, zero-expected-profit entry can take place even though Condition EP is satisfied.

Example 2 Consider a two-type economy in which both types have the same preferences represented by $u(q, t) \equiv q - q^2 - t$, but different costs such that $c_1 < 1 < \bar{c}_1 < c_2$; thus Condition EP is strictly satisfied. Both types are indifferent between not trading and trading the quantity $1 - c_1$ at unit price c_1. An entrant offering the contract $(1 - c_1, c_1(1 - c_1))$ earns zero expected profit if type 1 accepts, and type 2 chooses not to trade with him.

Even under strict single-crossing, it is still possible that the expected profit be exactly zero on any layer $q_i - q_{i-1}$. A simple and natural way to rule out this knife-edged situation is to assume that the buyer's preferences are strictly convex. Indeed, under this additional assumption, the inequalities (3) directly imply that the expected profit from any quantity layer $q_i - q_{i-1}$ is strictly negative whenever $q_{i-1} < q_i$. Again, this assumption is tight. Indeed, the following example shows that, when the buyer's preferences are only weakly convex, zero-expected-profit entry can take place even though types are ordered according to the strict single-crossing condition and Condition EP is satisfied.

Example 3 Consider, in line with Samuelson (1984), Myerson (1985), and Attar, Mariotti, and Salanié (2011), an economy in which a divisible good is traded, subject to a capacity constraint $q \in [0, 1]$. Every type i has linear preferences represented by $u_i(q, t) \equiv \overline{c}_i q - t$, where $\overline{c}_i$ is strictly increasing in i. Under this highly nongeneric assumption, strict single-crossing is satisfied and Condition EP is satisfied with equality for each type. Suppose now that the entrant offers a menu of contracts $\{(q_1, t_1), \ldots, (q_I, t_I)\}$ with strictly positive quantities q_i that are nondecreasing in i and transfers t_i such that $t_i - t_{i-1} = \overline{c}_i(q_i - q_{i-1})$. Any such allocation yields zero expected profit for the entrant and features strict gains from trade for types $i > 1$. The intuition is that Condition EP rules out gains from trade for any type i on the quantity layer $q_i - q_{i-1}$ but not necessarily, for $i > 1$, on the inframarginal quantity layers $q_j - q_{j-1}$, $j < i$. Hence, whereas strictly profitable entry is ruled out by Theorem 1, zero-expected-profit entry is possible, in many different ways, if every type i accepts to trade (q_i, t_i), even though she could as well choose to trade (q_{i-1}, t_{i-1}).

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