# THE WISDOM OF THE CROWD AND HIGHER-ORDER BELIEFS

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ABSTRACT: The classic wisdom-of-the-crowd problem asks how a principal can "aggregate" information about the unknown state of the world from agents without understanding the information structure among them. We propose a new simple procedure "*population mean based aggregation*" to achieve this goal. It only requires eliciting agents' beliefs about the state, and also eliciting some agents' expectations of the average belief in the population. We show that this procedure fully aggregates information: in an infinite population, it always infers the true state of the world. The procedure can accommodate correlation in agents' information, misspecified beliefs, any finite number of possible states of the world, and only requires very weak assumptions on the information structure.

KEYWORDS: aggregating beliefs, higher-order beliefs, wisdom of the crowd.

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## 1. Introduction

A long-held belief, starting from at least Aristotle if not earlier, is that of the 'wisdom of the crowd.' This refers to the idea that even though some event may be uncertain, and the information each individual has may be noisy; the 'aggregate' of individual beliefs is accurate. Such aggregation obviously has large social and private value, especially when it concerns important social or economic events. As a consequence, there has been substantial effort toward improving existing institutions (e.g. polling), designing and implementing new methods (e.g. prediction markets). At a theoretical level, then, it is important to understand the limits of such an exercise: without specific assumptions on the nature of the information agents have, how can we aggregate it, and how much can we learn from it? In short, how wise is the crowd, and how can we tap into its wisdom?

To fix ideas, imagine a population of interest. There is an event whose realization is uncertain. Individual agents each see a signal about the event which is informative of the state. Agents also understand the informational environment, i.e. they understand both what this signal implies for their own beliefs, and the implied distribution of other agents' beliefs. Even assuming away the difficulties (both logistical and strategic) of eliciting the relevant information from the population, the question we address is: does there exist a general procedure to aggregate individuals' information without knowledge of the information structure among them?

This paper provides an answer by constructing a simple procedure we name *Population Mean Based Aggregation* (herein PMBA). Our procedure involves eliciting all agents' beliefs about the uncertain event. It also requires eliciting, from some agents, their expectation about the average of others' beliefs. We show that, under very weak conditions, this procedure *fully aggregates* the information among agents in a large population. In other words, using information (solely about agents' own beliefs, and their expectation of the average of others' beliefs) elicited from agents, but with no other knowledge of the underlying information structure, it is possible in theory to fully learn the underlying state.

To situate our results, consider the following two important results from the literature (we defer a fuller survey of the literature to Section 5). The first paper, Arieli, Babichenko, and Smorodinsky (2017), considers a setting where the agent's signals are conditionally i.i.d. and the principal can elicit their first-order beliefs, i.e. beliefs about the event. They show that without further (very strong) assumptions about the signal structure, the principal cannot learn the state of the world, even with an infinite number of agents in the population: in the terminology of econometrics, there is an "identification problem."

The second paper, Prelec, Seung, and McCoy (2017) posits that the principal can also elicit agents' second-order beliefs, i.e., agents' beliefs about the distribution of beliefs in the population. They construct a mechanism ("surprisingly popular", herein SP) which elicits and uses second-order beliefs. They show that the ability to use second-order beliefs can vastly improve performance relative to aggregation based only on first-order beliefs, especially in the case of binary states, i.e. where there are only two possible states of the world. We discuss the connection to the SP mechanism in detail in Section 5.1. At a high-level the connection can be described thus: our procedure shows the strength of this approach, explaining that SP is applicable in the case of binary states even more broadly than the existing results suggest. It also explains how to more fruitfully use higher-order beliefs when there are multiple states (indeed, Prelec, Seung, and McCoy (2017) themselves point out the difficulties of extending the SP mechanism when there are three or more states, we recap the issues there).

Our point of departure is the idea of the latter paper, i.e. what if one could elicit agents' higher-order beliefs? Our main result considers general environments with a finite number of states of the world, and a large number of agents. Agents observe signals according to an information structure that is unknown to the aggregation procedure, but understood among them. We show that, under fairly general conditions, full aggregation can be obtained from the population distribution of first-order beliefs, and by eliciting information about second-order beliefs from a few agents, in particular some agents' expectations about the population average. Most strikingly, if the event is binary, a leading application of our setting, one needs the second-order beliefs of *only two agents* whose first-order beliefs disagree!

As a different application, recall that the highly influential "Bayesian Truth Serum" (BTS) of Prelec (2004) uses agents' higher-order beliefs to incentivize truthful reporting on surveys. To understand the contribution of BTS, recall that when there is an objective truth/ realization of states that is revealed to the principal, a proper scoring rule can be used to incentivize truthful reporting of beliefs by an agent about the state. BTS shows how to incentivize truthful reporting even when the underlying state is subjective/ not revealed to the principal—using agents' higher-order beliefs. We show in what follows that our PMBA procedure can be analogously used to incentivize truthful reporting of agents' beliefs when the underlying state is unknown/ unverifiable to the principal.

While our the present paper is theoretical, we note that the idea of eliciting agents' higher-order beliefs and using this information has been successfully implemented in practice. Indeed, Prelec, Seung, and McCoy (2017) report experimental evidence that such beliefs can be elicited and used, and there is a substantial literature describing implementations and variations of BTS (we describe this and other related literature in more detail in Section 5).

The basic idea of our mechanism is intuitive and can be described verbally, especially in the case of binary states. For simplicity, suppose every agent sees a conditionally i.i.d.

signal about the unknown state according to an information structure that is known to them but unknown to the principal. Now suppose the principal elicits, from each agent, their belief about the state. By a law of large numbers, the population average belief must equal the expected belief in that state. If the principal knew the expected belief in each state, she would be done: she could simply compare the elicited average to the expected belief in each state.[1] However she does not know this mapping from state to expected beliefs. To this end, suppose she elicited, from two agents with different beliefs about the states, their expectation of the population average belief. By the law of iterated expectations, each must report the weighted average of the expected belief conditional on state, weighted by their (different) beliefs about the likelihood of the two states. This presents the principal a simple system of two linear equations in two unknowns: thus the expected beliefs in each state can be uniquely recovered from the information elicited from these two agents! We show in what follows that this basic idea can be adapted with more states, limited correlation among agents, misperceptions among agents about other agents' beliefs etc.

The plan for the rest of the paper is as follows. In Section 2 we introduce the model and notation and formally present our PMBA mechanism. Section 2.2 restricts to the case of binary states, since our sufficient conditions (and the mechanism) are easier to describe in that case—further that case is perhaps the leading candidate to be used in practice. Section 2.3 gives the general mechanism for an arbitrary finite number of states. Section 3.1 considers an extension to a setting where agents have individually incorrect beliefs about the distribution of beliefs of others, and shows how to extend PMBA to such settings. It also discusses how PMBA can be used to elicit beliefs truthfully. Section 4 attempts to understand the full power and limitations of aggregation when eliciting higher-order beliefs. Of course the PMBA shows that with an infinite population, access to agents' first-order and (a few agents') second-order beliefs suffice. So, we now consider the case of a finite populations. We show that if the principal can extract the full hierarchy of agents' higher order beliefs, he can learn the "full information posterior", i.e. the posterior of an omniscient agent who shares the agents' prior and sees all their individual signals, despite the principal not knowing the prior and only collecting information about posteriors. Conversely, we show that if the principal only sees a finite prefix of the hierarchy (e.g. only the first $k$ beliefs) of a population then there is an "identification problem," i.e. there exist two different information structures that could both result in this exact realization. In short, higher-order beliefs are not useful unless the entire hierarchy can be elicited. Section 5 discusses the related literature in further detail and concludes. In

[1] As long as signals are informative, it can be shown that these expected beliefs must be different in the two states.

particular, Section 5.1 studies the implications on the Surprisingly Popular (SP) mechanism of Prelec, Seung, and McCoy (2017): we demonstrate both how our analysis implies that the SP mechanism is more generally applicable than previously understood, and also how our mechanism is applicable when SP is not.

## 2. Population-Mean Based Aggregation

In this section, we introduce our core model; describe our Population-Mean Based Aggregation procedure (herein PMBA) and prove its validity.

### 2.1. *General Model and Notation*

There is an infinite population of agents. To this end, let $N = \{1, 2, \ldots\}$ be a countably infinite set of agents, with $i$ denoting a generic agent. There are a finite set $L$ of states of the world $\Omega = \{\omega_1, \omega_2 \ldots \omega_L\}$. The space of feasible beliefs over $\Omega$, i.e. $\Delta(\Omega)$, is the $L$ dimensional simplex $\Delta^L$.

Each agent $i$ observes a signal in a set $S$ with associated sigma field $\mathcal{F}$. To allow for correlation, let us denote the common prior over $\Omega \times S^N$ by $P$. Given that an agent observes some signal $s \in S$, her posterior belief that the state is $\omega$ is given by $P(\omega|s)$. Let us denote the posterior beliefs of agent $i$ by $\mu_i \in \Delta^L$ (suppressing the dependence on the signal), and $\tilde{\mu}_i$ as the associated random variable. Define

$$\overline{\mu}_i(\omega) = \mathbb{E}[\tilde{\mu}_i|\omega].$$

In words, $\overline{\mu}_i(\omega)$ is the expected belief of agent $i$ in state $\omega$.

Finally, define $\overline{\mu}(\omega) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \overline{\mu}_i(\omega)$. This is the expected population average belief in state $\omega$.

Given a vector $x \in \Delta(\Omega)$, we will let $x_\omega$ refer to the component corresponding to $\omega$; i.e. $\mu_{i,\omega}$ is the belief of agent $i$ that the state is $\omega$, $\tilde{\mu}_{i,\omega}$ is the random variable of agent $i$'s belief that the state is $\omega$ etc.

### 2.2. *Binary States*

In this section, we consider the case of binary states, i.e. $L = 2$. This mechanism will form the core of our general analysis.

We denote by $P^S_{i,\omega}$ the induced marginal distribution over signals for agent $i$ in state $\omega$. We also abuse notation so that $\mu_i$ represents agent $i$'s belief that the state is $\omega_1$: since there are only two states, their belief that the state is $\omega_2$ is $1 - \mu_i$.

**Assumption 1.** *We make the following assumptions on $P$:*

1. *Imperfectly informed agents: for each agent $i$ we have that $P^S_{i,\omega}$ and $P^S_{i,\omega'}$ are mutually absolutely continuous with respect to each other.*

(2) *Limited correlation: For every $\varepsilon > 0$, there exists a finite $n(\varepsilon)$ such that conditional on the state, for any agent $i$, all but at most $n(\varepsilon)$ agents' signals are $\varepsilon$-independent of agent $i$'s signal. Formally:*

$$\forall \varepsilon > 0, \exists n(\varepsilon) \in \mathbb{N} \; s.t. \; \forall i, \exists N_i \subseteq N, |N_i| \leq n(\varepsilon),$$
$$\forall j \in N \backslash N_i, \forall E, E' \in \mathcal{F}, \forall \omega : |P(\tilde{s}_{i,\omega} \in E, \tilde{s}_{j,\omega} \in E' | \omega) - P^S_{i,\omega}(E) P^S_{j,\omega}(E')| \leq \varepsilon.$$

This assumption may seem a little dense so some discussion is useful. The first condition is just a normalization: there is no point aggregating the opinions of agents when some agent may be perfectly informed, so we rule out this possibility. Next, we need an assumption regarding the correlations between agents' signals. Intuitively, assuming conditionally i.i.d. signals is overly strong and does not capture the idea that agents' information may be correlated. However, we cannot allow for arbitrary correlation—for example, if all agents' information is fully correlated, then effectively there is one unique opinion in the population and no further aggregation is possible. Our notion allows for "enough independence," roughly guaranteeing that for any agent, most other agents' signals are approximately independent conditional on state. This ensures that a law of large numbers holds, so that the population average belief is a deterministic function of state, i.e. in the notation of our model, in any state $\omega$, a.s. for any realizations $\mu_i$ of beliefs of agents $i \in N$; we have that

$$\hat{\mu} \equiv \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mu_i = \overline{\mu}(\omega).$$

As we will see in what follows, our PMBA procedure requires that the population average beliefs are different in the two states, i.e. $\overline{\mu}(\omega_1) \neq \overline{\mu}(\omega_2)$. This is a very weak condition, but one that one may nevertheless worry about: in principle different distributions of beliefs may nevertheless have the same expectation. The following assumption delivers this: it requires that all agents individually are at least minimally informed, formally defined with respect to the total variation metric. This is also akin to a normalization, it simply ensures that we do not have infinitely many arbitrarily uninformed agents.

To this end, we need some more notation: let $G_{i,\omega}$ denote the induced distribution of private beliefs of agent $i$ in state $\omega$. We will abuse notation and also let it denote the cumulative distribution function of the belief of agent $i$ in state $\omega$.

**ASSUMPTION 2** (Minimal Information)**.** *We assume that there exists $\delta > 0$ such that for any agent $i$ the total variation distance between $G_{i,\omega_1}$ and $G_{i,\omega_2}$ is uniformly bounded below by $\delta$, i.e.*

$$\exists \delta > 0 \; s.t. \; \forall i : \sup_{E \subseteq [0,1]} |G_{i,\omega_1}(E) - G_{i,\omega_2}(E)| \geq \delta.$$

PROCEDURE 1. Population-Mean Based Aggregation, 2 states

(1) Elicit, from each agent $i \in N$, her belief $\mu_i$. Calculate $\hat{\mu}$ defined as:

$$\hat{\mu} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mu_i$$

(2) Select 2 agents, A and B, such that $\mu_A \neq \mu_B$. Two such agents must exist almost surely.

Elicit from each of these agents their expectation of the average posterior beliefs in the population, i.e., denoting $s_i$ as the signal seen by agent $i$, elicit for each $i = A, B$:

$$\alpha_i = \lim_{n \to \infty} \mathbb{E}\left[\frac{1}{n}\sum_{j=1}^{n} \tilde{\mu}_j \middle| s_i\right].$$

(3) Calculate $\overline{\mu}(\omega_1), \overline{\mu}(\omega_2)$ as:

$$\begin{pmatrix} \overline{\mu}(\omega_1) \\ \overline{\mu}(\omega_2) \end{pmatrix} = \begin{pmatrix} \mu_A & 1-\mu_A \\ \mu_B & 1-\mu_B \end{pmatrix}^{-1} \begin{pmatrix} \alpha_A \\ \alpha_B \end{pmatrix} \tag{1}$$

Note that the right hand side is elicited from agents directly; and since $\mu_A \neq \mu_B$, the middle matrix is invertible.

(4) Recover the state by comparing $\hat{\mu}$ calculated in step 1, with $\overline{\mu}(\cdot)$ calculated above. The state of the world is $\omega$ s.t. $\hat{\mu} = \overline{\mu}(\omega)$.

The first main result of our paper shows the power of the PMBA procedure described in Procedure 1.

**THEOREM 1.** *Suppose the information structure among agents satisfies Assumptions 1 and 2. Then* population mean based aggregation *procedure (Procedure 1) correctly recovers the true state of the world a.s.*

A formal proof is in Appendix A. Here we want to discuss how the various Assumptions play a role in the result. As we pointed out earlier, Part (1) of Assumption 1 is mainly a normalization. The key processing step in this procedure is Step 3. By the law of total probability, an agent $i$ with beliefs $\mu_i$ will report, in step (2),

$$\alpha_i = \mu_i \overline{\mu}(\omega_1) + (1-\mu_i)\overline{\mu}(\omega_2).$$

Our procedure simply inverts this relationship to learn the unknown mapping from $\omega$ to $\overline{\mu}(\omega)$., using equation 1.

Step (4) then recovers the state by comparing the calculated $\overline{\mu}(\cdot)$ to the elicited $\hat{\mu}$. Part (2) of Assumption 1 ensures that an appropriate law of large numbers holds so that in state $\omega$, $\hat{\mu} = \overline{\mu}(\omega)$, almost surely. Assumption 2 is used to guarantee that $\overline{\mu}(\omega_1) \neq \overline{\mu}(\omega_2)$. Therefore, almost surely, the elicited $\hat{\mu}$ will match $\overline{\mu}(\omega)$ for a unique state, and the procedure successfully concludes. Taken together, therefore, these imply our result.

To be clear, the model and result of this section are explicitly written for a common prior model where the agents perfectly understand the signal structure amongst themselves, only the principal does not. In applied contexts, this kind of knowledge of each others' information among agents may seem daunting or even implausible.

To that end, we should firstly point out that our mechanism does not require agents have an exact understanding of the distribution of signals among fellow agents. Observe that it is sufficient for each agent to perfectly understand the aggregate implications, that is to say, the average population beliefs in each state, $\overline{\mu}(\cdot)$ since that is all they need to know (other than their own beliefs) to calculate the information that we elicit. This is a substantially lower informational requirement. Indeed in the example in Section 3.2, there we show how even less demanding information may be elicited (e.g. ask most agents what state they think is more likely).

However, even this may seem high—perhaps agents don't perfectly understand even the aggregate implications of the model—for example, each agent has noisy information about the true $\overline{\mu}(\cdot)$, rather than exact knowledge of it. In Section 3.1, we show how a modification of our mechanism will continue to allow for population aggregation in such a setting.

### 2.3. $L > 2$ *States*

We now consider the case of $L > 2$ states. Intuitively, we generalize the "matrix inversion" ideas of the previous subsection to propose a PMBA procedure which works for $L$ states. However having more than 2 states introduces some difficulties: notably, the assumptions which guarantee that this process works become slightly more demanding. Assumption 1 continues to apply as written, observe that nothing in that assumption was specific to the case of binary states.

However, a direct generalization of Assumption 2 to more than 2 states is generally not sufficient to assure that state-dependent averages are unequal across states. So we assume the $L$ state generalization of its implication directly.

**Assumption 3.** *We assume that $P$ is such that for any two states $\omega$ and $\omega'$, we have that $\overline{\mu}(\omega) \neq \overline{\mu}(\omega')$.*

Note that while Assumption 3 is not directly in terms of the primitives of our model ($P$), it nevertheless is a property that is generically satisfied.

Finally, in the 2−state case, we were guaranteed the a.s. existence of two agents who had different beliefs. The appropriate generalization turns out to be the existence of $L$ agents whose beliefs constitute a full rank $L \times L$ matrix so that the analog/ generalization of (1) in step (3) can be carried out. The following assumption guarantees this:

**ASSUMPTION 4.** *We assume that $P$ is such that for any agent $i$, the support of beliefs supp$(\tilde{\mu}_i)$ contains $L$ distinct points such that those $L$ beliefs, viewed as $L$-dimensional vectors, constitute a set of full rank. Alternately and equivalently, we require that the convex-hull of the support has an interior relative to $\Delta^L$.*

One might wonder why such a condition was not required in the case of binary states. The answer is that when $L = 2$, this condition simply reduces to requiring that the support of the agents' beliefs has at least two points in any state. Part (1) of Assumption 1 however implies that the support of agents' beliefs is the same across states. If this support was degenerate at a single point, we would clearly violate Assumption 2. Note that with $L > 2$, there clearly exist settings where which violate Assumption 4 despite satisfying Assumptions 1 and 2: for instance, consider a setting where $L = 3$ so that the simplex is a triangle, but there are only two possible signals so that there only two induced beliefs.

We are now in a position to describe our main procedure, Population Mean Based Aggregation (PMBA).

PROCEDURE 2. Population Mean Based Aggregation, $L$ states

(1) Elicit, from each agent $i \in N$, her belief $\mu_i$. Calculate $\hat{\mu}$ defined as:
$$\hat{\mu} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mu_i$$
(2) Select $L$ agents $I = \{i_1, i_2, \ldots, i_L\}$ such that the $L \times L$ matrix $\boldsymbol{\mu} = (\mu_{i_1}^T, \mu_{i_2}^T, \ldots, \mu_{i_L}^T)^T$ is full rank.
(3) Let $s_i$ be the signal seen by agent $i$. Elicit from each of these agents their expectation of the average posterior beliefs in the population, i.e., elicit for each $i \in I$:
$$\alpha_i = \lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n}\sum_{j=1}^{n} \tilde{\mu}_j \middle| s_i\right].$$
Denote $\boldsymbol{\alpha} = (\alpha_{i_1}^T, \alpha_{i_2}^T, \ldots, \alpha_{i_L}^T)^T$.
(4) Solve
$$\bar{\boldsymbol{\mu}} = \boldsymbol{\mu}^{-1}\boldsymbol{\alpha} \tag{2}$$
Note that the right hand side is elicited from agents directly and therefore $\bar{\boldsymbol{\mu}}$ can be recovered. The state $\omega$ is the row of the recovered $\bar{\boldsymbol{\mu}}$ that equals the calculated $\hat{\mu}$.

**THEOREM 2.** *Suppose $L > 2$ and the common prior $P$ satisfies Assumptions 1, 3 and 4. Then the PMBA procedure (Procedure 2) recovers the true unknown state of nature $\omega$ almost surely.*

## 3. Extensions

In this section, we show how to extend the basic PMBA to "more realistic" settings. In Section 3.1 we show a variant of the procedure that works even if individual agents have potentially incorrect information about the aggregate distribution of signals. In Section 3.2, we recognize that in practice, eliciting beliefs from agents is difficult— we show that a variant of the procedure can aggregate information if we only elicit "simple" information (e.g. report which state you think is more likely) from most agents, and elicit more fine grained from only a small number of agents. Both extensions are displayed for the case of 2 states of the world.

### 3.1. *Aggregation in Misspecified Information Settings*

In what follows, we show that even if the agents have limited and individually incorrect knowledge about the aggregate information structure, nevertheless similar ideas can be used to aggregate population information. That is, eliciting first-order beliefs from all, and a second-order statistic from a subgroup of agents will be sufficient to learn the state. In the interests of brevity, we consider the case of two states, the extension to $L$ states is analogous to the previous section.

We maintain the assumptions on the true signal generating process $P$ as in Section 2.2, i.e. Assumptions 1 and 2. Instead of assuming common knowledge of $P$, we assume that agents' information about $P$ is misspecified—agent $i$ believes that the population means in each state are denoted by $(\alpha_i^{\omega_1}, \alpha_i^{\omega_2}) \in [0,1]^2$. We make the following assumption on the relation between agents' beliefs and the true state-dependent population means.

**Assumption 5.** *We make the following assumption regarding the agents' knowledge about $P$: For each agent $i$, and each state $\omega \in \Omega$ we have that:*

$$\alpha_i^{\omega} = \overline{\mu}(\omega) + \zeta_i^{\omega}$$

*where $\zeta_i^{\omega}$ is a mean-zero noise term that is conditionally independent across agents (independent conditioned on state and agents' signals). Further, we assume that the support of $\zeta_i^{\omega}$ is such that the random variables $\alpha_i^{\omega_1}$ and $\alpha_i^{\omega_2}$ have disjoint support.*

Let us briefly discuss Assumption 5. One implication of this is that each agent understands that conditional on the state her expectation of the population mean is independent of her first-order belief. The disjoint assumption on supports is necessary keeping in mind that agents are fully Bayesian in their inferences, they simply have misspecified knowledge. To be precise, this assumption avoids that their expectation for population average in state $\omega_1$ is larger than in state $\omega_2$ which would contradict Bayes rationality.

We can now state our aggregation result in the limited information setting.

PROCEDURE 3. Limited Information Population Mean Based Aggregation

(1) Elicit, from each agent $i$, her belief $\mu_i$. Calculate $\hat{\mu}$ defined as:

$$\hat{\mu} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mu_i$$

(2) Elicit from all agents their interim expectation of the average posterior belief over all agents— denote this as $\alpha_i$ for agent $i$.

Let group $A$ be the set of agents $i$ such that $\mu_i \leq \hat{\mu}$, and $B$ all the other agents. Calculate the average of the $\alpha_i$'s in each group, $A, B$. For example, without loss, rename agents so that $A = \{1, 2, 3, \ldots\}$. Then define

$$\alpha_A = \lim_{n\to\infty} \frac{1}{n} \sum_{i\leq n} \alpha_i,$$

$$\mu_A = \lim_{n\to\infty} \frac{1}{n} \sum_{i\leq n} \mu_i.$$

Calculate $\mu_B, \alpha_B$ analogously.

(3) Calculate $\overline{\mu}(\omega)$ as

$$\begin{pmatrix} \overline{\mu}(\omega_1) \\ \overline{\mu}(\omega_2) \end{pmatrix} = \begin{pmatrix} \mu_A & 1-\mu_A \\ \mu_B & 1-\mu_B \end{pmatrix}^{-1} \begin{pmatrix} \alpha_A \\ \alpha_B \end{pmatrix} \tag{3}$$

Recover the state by comparing $\hat{\mu}$ calculated in step 1, with $\overline{\mu}(\omega), \omega \in \Omega$ calculated above.

**THEOREM 3.** *Suppose the true signal distribution process satisfies Assumptions 1 and 2, and that the agents' knowledge regarding the information structure satisfies Assumption 5. Then* Limited Information Population Mean Based Aggregation *(Procedure 3) correctly recovers the true state of the world almost surely.*

Theorem 3 establishes that indeed the conceptual idea of second-order elicitation successfully overcomes the limited understanding of agents' regarding the private belief distributions of their peers. Compared to the PMBA aggregation under complete knowledge of the information structure, the two infinite groups $A$ and $B$ play the role of the two agents in the PMBA.

Thus moving to a more realistic assumption of an incomplete and individually incorrect understanding of the information structure necessitates the elicitation of the second-order statistics from large groups. Thus we are relying on the crowd for both first and second-order statistic. The main issue is that while individual agents' understanding of the population averages is incorrect. However, by a law of large numbers we can appropriately average these individual agents' reports over two groups $A$ and $B$. The average first order belief, and average report of population average, of agents in each group, will serve as the analogs of the two individuals in the baseline PMBA procedure (Procedure 1).

The basic intuition is that even though individual agents are misinformed about the distribution of fellow agents' beliefs, these errors are not systematic. One can show, therefore, that, the constructed $\alpha_A, \alpha_B$ for these groups $A$ and $B$ would be the same, as agents in this setting with beliefs $\mu_A, \mu_B$ respectively who also were correctly informed about the distribution of agents' beliefs. Assumption 5 makes this possible.

### 3.2. *Eliciting Guesses instead of Beliefs*

In this section we discuss how our PMBA mechanism can be adjusted to environments where not beliefs but rather finite guesses are elicited, e.g., which state is more likely rather than what is the probability of state $\omega = 1$. Consider a binary state space and an information structure that satisfies Assumptions 1 and 2. The action based PMBA procedure is defined as follows.

PROCEDURE 4. Action based PMBA, 2 states

(1) Elicit, from each agent $i \in N$, the state $\omega \in 0, 1$ she considers more likely, $a_i \in 0, 1$. If her belief equals 0.5 have her report $a_i = 1$. Calculate $\hat{\alpha}$ defined as:

$$\hat{\alpha} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i$$

(2) Select 2 agents, A and B, such that $a_A \neq a_B$. Elicit from each of these agents $i = A, B$ their belief $\mu_i$ and their expectation of the average report in the population, i.e., denoting $s_i$ as the signal seen by agent $i$, elicit for each $i = A, B$:

$$\alpha_i = \lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n}\sum_{j=1}^{n} a_j \middle| s_i\right].$$

(3) Calculate $\bar{\alpha}(\omega_1), \bar{\alpha}(\omega_2)$ as:

$$\begin{pmatrix} \bar{\alpha}(\omega_1) \\ \bar{\alpha}(\omega_2) \end{pmatrix} = \begin{pmatrix} \mu_A & 1-\mu_A \\ \mu_B & 1-\mu_B \end{pmatrix}^{-1} \begin{pmatrix} \alpha_A \\ \alpha_B \end{pmatrix} \tag{4}$$

Note that the right hand side is elicited from agents directly; and since $\alpha_A \neq \alpha_B$, the middle matrix is invertible.

(4) Recover the state by comparing $\hat{\alpha}$ calculated in step 1, with $\bar{\alpha}(\cdot)$ calculated above. The state of the world is $\omega$ s.t. $\hat{a} = \bar{a}(\omega)$.

For the action based PMBA procedure to be viable we require the existence of two agents $A, B$ with differing first-order guesses, $a_A \neq a_B$. Assumptions 1 and 2 are not sufficient to assure this. The following assumption however guarantees the existence of two such agents.

**ASSUMPTION 6** (No Herding)**.** *We assume that there exists $\delta > 0$ such that for any agent $i$ we have that the probability agent $i$ assigns to state $\omega_j$ conditional on her signal $s_i$ is larger than half with a probability of at least $\delta$, for $\omega_j = \omega_1, \omega_2$.*

We can now present our result on aggregation in binary guess environment.

**PROPOSITION 1.** *Consider a binary state space. Suppose that the information structure satisfies Assumptions 1, 2 and 6. Then the* Action Based PMBA *procedure (Procedure 4) correctly recovers the true state of the world a.s.*

The result follows along the same lines as Theorem 1.

### 3.3. *Incentivizing Truthful Reporting*

Recall that when there is a verifiable state that is or will be known to the principal, the principal can truthfully elicit agents' beliefs about the state by paying them using a proper scoring rule. A proper scoring rule pays agents as a function of reported beliefs and true state so that for any agent, truthful reporting of beliefs is a dominant strategy. For example classic scoring rules such as the Brier score (Brier, 1950) or the logarithmic scoring rule handle make truthful reporting a dominant strategy for expected utility maximizing agents.[2]

When the state is not verifiable/ observed to the principal, proper scoring rules are not applicable. However, the principal may nevertheless be interested in ensuring that agents are incentivized to report their beliefs truthfully. A leading example from Prelec (2004) is the case of subjective surveys/ evaluations of a new product: the principal (e.g. a marketer) might wish to incentivize truthful reporting by the agents, but there is no "ground truth" on which to base the rewards. The basic idea of BTS is that agents' beliefs also have implications on their second-order beliefs (i.e. how they think the population will evaluate it). The population averages are indeed verifiable, and therefore can be used to incentivize agents. A similar idea works here, we describe it loosely for completeness:

(1) Elicit beliefs and second-order beliefs from agents as prescribed in PMBA to learn the true state $\omega$ (which exact variant of PMBA is used, and what is elicited, depends on the setting).
(2) For all agents who were asked for second-order beliefs, pay them based on a proper scoring rule as a function of their expected population average beliefs, and the true calculated population average beliefs.
(3) For all agents, pay them using a proper scoring rule as a function of their beliefs and the state $\omega$ recovered by PMBA.

[2]There is a large literature proposing more robust scoring rules for various settings, see e.g. Karni (2009) for an example.

Observe that under the maintained assumptions of correctness of the corresponding PMBA mechanism, truthful reporting will be a perfect Bayesian equilibrium among agents.

## 4. Aggregation with Finite Populations

So far, we have discussed the possibility of aggregating information with an infinite population. In practice, of course, populations are finite. The procedures given above will correctly identify the true state with high probability in large populations. For instance, in our baseline model, even with a large but finite population, the average belief in the population concentrates around the expected belief conditional on the true state. As long as the expected beliefs conditional are sufficiently different across the states, an appropriately modified procedure will recover the true state of the world with high probability.

Nevertheless, at a theoretical level, one may wonder if, with a finite population, there is any value to eliciting higher-order beliefs, ignoring the difficulties of such elicitation in practice. The benchmark we use to assess this is one of a "full information posterior:" i.e. can we, without knowledge of the underlying information structure $P$, and eliciting solely agents beliefs (and higher-order beliefs), nevertheless reach the same posterior beliefs as an omniscient agent who knew $P$ and directly observed all agents' signals? Our previous results answered this in the positive showed that for the case of an infinite population (under the maintained assumptions), the output of the PMBA identifies the degenerate belief on the true state.

In this section we answer these questions primarily in the negative. First we show that with a finite population, and elicitation of the entire hierarchy of beliefs, an agnostic mechanism can learn the prior and signals of each agent. This result is essentially a straightforward corollary of the results of Mertens, Sorin, and Zamir (2015). However, elicitation of the entire hierarchy is obviously impractical in applications.

By contrast, we show that, for elicitation up to any finite-order of beliefs, there is an identification problem: there exist information structures where the exact same finite hierarchy can realize among agents in both states.

### 4.1. *Eliciting the entire hierarchy of beliefs*

Consider a finite set of agents $\{1, 2, ..., N\}$ with a common prior $\mu$ defined on a finite set $\Omega \times S$ (where $S = \times_{i=1}^{N} S_i$ is set of their signal profiles).[3] The aggregator does not know $\mu$ but can ask the agents to report their higher-order beliefs. In this section, we show that the aggregator can effectively elicit the full information posterior, provided that she can

[3] We are overloading $\mu$ relative to previous sections, but this should be clear in context.

ask the agents to report their entire hierarchies of beliefs and each hierarchy of beliefs uniquely identifies a signal of an agent.

To this end we recall the standard formulation of higher-order beliefs due to Mertens and Zamir (1985). Denote by $s_i^k$ the $k$th-order belief of agent $i$ over $\Omega$. For instance, $s_i^1 = \text{marg}_{\Omega}\mu\left(\cdot|s_i\right)$,

$$s_i^2\left(\omega, s_{-i}^1\right) = \mu\left(\left\{\left(\omega', s'_{-i}\right) : \omega' = \omega \text{ and } s'^1_{-i} = s^1_{-i}\right\}|s_i\right),$$

and so on. Denote by $\tilde{s}_i$ the hierarchy of beliefs induced by signal $s_i$, i.e., $\tilde{s}_i = \left(s_i^1, s_i^2, ...\right)$. Let $\tilde{S}_i$ be the set of hierarchies of beliefs induced from $S_i$ and $\tilde{\mu}$ the distribution induced by $\mu$ on $\Omega \times \tilde{S}$. Moreover, by Mertens and Zamir (1985), each $\tilde{s}_i$ induces a belief $\tilde{\pi}_i\left(\tilde{s}_i\right)$ over $\Omega \times \tilde{S}_{-i}$ where $\tilde{S}_{-i} = \times_{j\neq i}\tilde{S}_j$.

**Theorem 4.** *Suppose that $\tilde{\mu}$ has a finite support and the agents report $\tilde{s} = \left(\tilde{s}_i\right)_i$. Then there exists a procedure which recovers the "pooled information" posterior on states, i.e. $\tilde{\mu}(\cdot|\tilde{s})$.*

### 4.2. *Eliciting higher-order beliefs up to finite order m*

We now explain that the above cannot be done if the aggregator only knows the reported beliefs up to order $m$. Our argument is adapted from the leading example from Lipman (2003).

We adopt his notation here for the ease of comparison. Suppose that there are two players and two states of nature $\{\omega_1, \omega_2\}$ as in our paper. The construction is easier to explain in terms of the standard partitional model of knowledge.

The model below considers 8 extended states: $\{(\sigma_l, k) : l \in \{1,2\} \text{ and } k \in \{1,2,3,4\}\}$. Interpret state $(\sigma_l, k)$ in this model as corresponding to $\omega_l$ realized for $l = 1,2$ and any $k$. In this example, Lipman considers the hierarchy of belief induced by common knowledge that player 1 assigns probability $2/3$ to $\omega_1$ and player 2 assigns probability $1/3$ to $\omega_1$. Such a type does not admit a common prior. However, Lipman shows that the model below, which has a uniform common prior, admits a state $(\sigma_1, 1)$ where the players have the same belief as that of the common knowledge type described above, up to any finite order $m$.

The prior for the players is the uniform distribution over the extended states, i.e.:

| | $(\sigma_1,4)$ | $(\sigma_1,3)$ | $(\sigma_2,2)$ | $(\sigma_1,1)$ | $(\sigma_2,1)$ | $(\sigma_1,2)$ | $(\sigma_2,3)$ | $(\sigma_2,4)$ |
|---|---|---|---|---|---|---|---|---|
| $\mu$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ |

Agents' information is identified with the following partitions:

$$\Pi_1 = \{\{(\sigma_1,4),(\sigma_1,3),(\sigma_2,2)\}\{(\sigma_1,1),(\sigma_2,1),(\sigma_1,2)\}\ \{(\sigma_2,3),(\sigma_2,4)\}\}$$
$$\Pi_2 = \{\{(\sigma_1,4),(\sigma_1,3)\}\{(\sigma_2,2),(\sigma_1,1),(\sigma_2,1)\}\{(\sigma_1,2),(\sigma_2,3),(\sigma_2,4)\}\}.$$

Note that each $\Pi_i(\sigma', k)$ is identified with a signal of player $i$. Observe that player 1 assigns probability one to $\omega_2$ at $\Pi_1(\sigma_2, 4)$ and player 2 assigns probability one to $\omega_1$ at $\Pi_2(\sigma_1, 4)$. Hence, each partition cell of each player induces a different second-order belief. In particular, $(\Pi_1(\sigma_1, 1), \Pi_2(\sigma_1, 1))$ is the only partition profile at which the second-order beliefs of both players are identical to $t$. Moreover, conditional on the reported second-order belief at $(\Pi_1(\sigma_1, 1), \Pi_2(\sigma_1, 1))$ (denoted as $(\Pi_1(\sigma_1, 1), \Pi_2(\sigma_1, 1))^2$), we have

$$\mu(\omega_1 | (\Pi_1(\sigma_1, 1), \Pi_2(\sigma_1, 1))) = \frac{1}{2}.$$

Now consider another model modified from the previous one by adding one additional state with the following prior:

| | $(\sigma_1, 4)'$ | $(\sigma_1, 3)'$ | $(\sigma_2, 2)'$ | $(\sigma_1, 1)'$ | $(\sigma_1, 1)$ | $(\sigma_2, 1)$ | $(\sigma_1, 2)$ | $(\sigma_2, 3)$ | $(\sigma_2, 4)$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mu'$ | $\frac{1}{20}$ | $\frac{1}{20}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $0$ | $\frac{1}{10}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ |

where $'$ is to indicate that a state is to the left of $(\sigma_1, 1)$ and it will be useful in generalizing the idea to eliciting $m$ orders of beliefs, for $m \geq 3$ in what follows. Agents' information is now given by the partitions:

$$\begin{aligned}
\Pi_1' &= \{\{(\sigma_1, 4)', (\sigma_1, 3)', (\sigma_2, 2)', (\sigma_1, 1)'\}, \{(\sigma_1, 1), (\sigma_2, 1), (\sigma_1, 2)\} \ \{(\sigma_2, 3), (\sigma_2, 4)\}\}; \\
\Pi_2' &= \{\{(\sigma_1, 4)', (\sigma_1, 3)'\}, \{(\sigma_2, 2)', (\sigma_1, 1)', (\sigma_1, 1), (\sigma_2, 1)\}, \{(\sigma_1, 2), (\sigma_2, 3), (\sigma_2, 4)\}\}.
\end{aligned}$$

Observe now that, conditional on the reported second-order belief at $(\Pi_1'(\sigma_1, 1), \Pi_2'(\sigma_1, 1))$ (denoted as $(\Pi_1'(\sigma_1, 1), \Pi_2'(\sigma_1, 1))^2$), we have

$$\mu'\left(\omega_1 | (\Pi_1'(\sigma_1, 1), \Pi_2'(\sigma_1, 1))^2\right) = 0.$$

Second, observe that

$$(\Pi_1'(\sigma_1, 1), \Pi_2'(\sigma_1, 1))^2 = (\Pi_1(\sigma_1, 1), \Pi_2(\sigma_1, 1))^2.$$

This means that the reported second-order beliefs at $(\Pi_1'(\sigma_1, 1), \Pi_2'(\sigma_1, 1))$ under $\mu'$ are the same as $(\Pi_1(\sigma_1, 1), \Pi_2(\sigma_1, 1))$ under $\mu$. The basic idea is that in $\mu'$, we "shift" the probability which $\mu$ assigns to $(\sigma_1, 1)$ to the additional state $(\sigma_1, 1)'$. This additional state helps us preserve the first-order belief of player 2 at $\Pi_2'(\sigma_1, 1)$ being still 1/3, while we decrease the probability $(\sigma_1, 4)$ to 0 to preserve the first-order belief of player 1 at $\Pi_1'((\sigma_1, 4)')$. This takes care of the states to the left of $(\sigma_1, 1)$. For states to the right, we double the probability of $(\sigma_2, 2)$, $(\sigma_2, 3)$, and $(\sigma_2, 4)$ so that the first-order belief of player 1 at $\Pi_1'(\sigma_1, 1)$ and the first-order belief of player 2 at $\Pi'$ are also preserved. Therefore, just eliciting the first two orders of beliefs, we cannot distinguish between the model corresponding to $\mu$ (under which the posterior would be that both states are equally likely), and $\mu'$ (under which the posterior would be that the state is $\omega_2$ for sure).

The construction for $m > 2$ is similar but more involved. The details are in Appendix C.1.

## 5. Related Literature and Conclusions

In what follows, we discuss related literature more closely (with the benefit of being able to explain some of the ideas more formally with our notation). We first discuss the SP mechanism of Prelec, Seung, and McCoy (2017) in some detail. Then we discuss other related literature and approaches. Finally, we conlcude.

### 5.1. *The Surprisingly Popular Mechanism*

We formally define the mechanism in our general environment, generalize existing results and highlight its limitation relative to PMBA. To start, consider the case of binary states and let the common prior P among agents satisfy Assumption 1 and Assumption 2. In our notation, the SP mechanism can be described follows:

PROCEDURE 5. Surprisingly Popular, 2 states

(1) Elicit, from each agent $i \in N$, her belief $\mu_i$. Calculate $\hat{\mu}$ defined as:

$$\hat{\mu} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mu_i$$

(2) Select one agent, $i$, and elicit her expectation of the average posterior beliefs in the population, i.e., denoting $s_i$ as the signal seen by agent $i$:

$$\alpha_i = \lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n}\sum_{j=1}^{n} \tilde{\mu}_j \middle| s_i\right].$$

(3) Compare the realized population average belief $\hat{\mu}$ with agent $i$'s expectation $\alpha_i$. If $\hat{\mu} < \alpha_i$ the state of the world is $\omega = 1$, otherwise it is $\omega = 0$.

The SP procedure is arguably even simpler than our PBMA procedure. Only one agent is required to report her expectation of the population average and no calculation is required. Nevertheless, SP recovers the truth in the binary state setting as the following result shows.

**PROPOSITION 2.** *Consider a binary state space. Suppose that the information structure among agents satisfies Assumptions 1 and 2. Then the* Surprisingly Popular *procedure (Procedure 5) correctly recovers the true state of the world a.s.*

Note that this result provides a stronger theoretical guarantee of the applicability of the Surprisingly Popular procedure: the existing results about SP require significantly stronger assumptions on primitives. We omit a formal proof. The result follows from arguments similar to those behind Theorem 2. Assumptions 1 and 2 assure that the population average is degenerate in both states and unequal. Further, by Lemma 3 the population average belief in state $\omega_1$ is strictly smaller than in state $\omega_2$. Hence agent $i$'s expected population average belief is strictly larger than the population average belief in state $\omega_1$, and strictly smaller than the population average belief in state $\omega_2$. Since the first-order belief of any agent $i$ is unequal to 0 and 1 by Assumption 1 the result follows. This argument can be easily extended to capture the limited knowledge setting considered in Section 3.1.

In summary, the SP procedure is extremely powerful in aggregating beliefs in the binary state setting. However, beyond two states it is not applicable without strong additional assumptions. The reason is that, in general, more than two states can be "surprisingly popular", leaving an indeterminacy in how to define the mechanism. Even when defining the SP procedure as e.g. picking the state that surpassed by most its expectation, the procedure does not achieve aggregation. This issue is discussed in Section 1.3 of the Supplementary Information of Prelec, Seung, and McCoy (2017). The authors impose an additional assumption (see e.g. Theorem 3), one of "diagonal dominance:" roughly that the posterior of a agent who thinks state $\omega$ is most likely also must place a higher posterior on $\omega$ than any voter voting for $\omega' \neq \omega$ places on $\omega$. Simple counter examples show that this assumption is necessary. We exhibit one below:

**EXAMPLE 1.** *Suppose there are three states $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and three signals $S = \{s_1, s_2, s_3\}$. All agents have the initial uniform prior. Agents have a uniform prior over signals and receive conditionally i.i.d. signals, conditional on state so that their posteriors can be described as:*

| | $P(\omega_1 \mid \cdot)$ | $P(\omega_2 \mid \cdot)$ | $P(\omega_3 \mid \cdot)$ |
|---|---|---|---|
| $s_1$ | 0.4 | 0.21 | 0.39 |
| $s_2$ | 0.45 | 0.54 | 0.01 |
| $s_3$ | 0.44 | 0.06 | 0.5 |

*i.e. the $i^{th}$ row and $j^{th}$ column is the posterior the agent places on $\omega_j$ on signal $s_i$ $P(\omega_j|s_i)$. Note that this violates the assumption of "diagonal dominance"— $P(\omega_1|s_2) > P(\omega_1|s_1)$, i.e. an agent who sees $s_2$ (or, indeed $s_3$) places a higher posterior on $\omega_1$ than an agent who sees signal $s_1$. However note that an agent who sees signal $s_i$ conditionally believes that $\omega_i$ is the most likely state.*

*To be clear note that this can be achieved by the following distribution of signal conditional on state (numbers rounded):*

| | $P(\cdot\|\omega_1)$ | $P(\cdot\|\omega_2)$ | $P(\cdot\|\omega_3)$ |
|---|---|---|---|
| $s_1$ | 0.31 | 0.259 | 0.433 |
| $s_2$ | 0.349 | 0.667 | 0.011 |
| $s_3$ | 0.341 | 0.074 | 0.556 |

*This results, by mechanical calculation, on the following average beliefs in the population:*

| | $\overline{\mu}(\omega_1)$ | $\overline{\mu}(\omega_2)$ | $\overline{\mu}(\omega_3)$ |
|---|---|---|---|
| $\omega_1$ | 0.431 | 0.436 | 0.422 |
| $\omega_2$ | 0.274 | 0.419 | 0.130 |
| $\omega_3$ | 0.295 | 0.145 | 0.447 |

*here the $i^{th}$ row and and $j^{th}$ column represent the average belief in the population that the state is $\omega_i$ when the true state is $\omega_j$.*

*Note that the resulting expected population average belief is:*

| | $\alpha(s_1)$ | $\alpha(s_2)$ | $\alpha(s_3)$ |
|---|---|---|---|
| $\omega_1$ | 0.429 | 0.434 | 0.427 |
| $\omega_2$ | 0.248 | 0.351 | 0.211 |
| $\omega_3$ | 0.323 | 0.215 | 0.362 |

*i.e. the column labeled $\alpha(s_i)$ is the expected population average belief of an agent seeing signal $s_i$.*

*More generally, we can summarize the set of "surprisingly popular" state(s) in each case as:*

| | $\omega_1$ | $\omega_2$ | $\omega_3$ |
|---|---|---|---|
| $\alpha(s_1)$ | $\omega_1, \underline{\omega_2}$ | $\omega_1, \underline{\omega_2}$ | $\omega_3$ |
| $\alpha(s_2)$ | $\omega_3$ | $\omega_1, \underline{\omega_2}$ | $\omega_3$ |
| $\alpha(s_3)$ | $\omega_1, \underline{\omega_2}$ | $\omega_1, \underline{\omega_2}$ | $\omega_3$ |

*i.e. the entry in the row $i$ corresponding to $\alpha(s_i)$ and column $j$ corresponding to state $\omega_j$ is the set of states that are surprisingly popular, in true state $\omega_j$, relative to the expected population average beliefs of an agent who received signal $s_i$. The underlined entry, if there are multiple, is the one which is surprising by the largest magnitude. For example, relative to an agent seeing $s_1$ in true state $\omega_1$, both $\omega_1$ and $\omega_2$ are surprisingly popular, but $\omega_2$ is the most surprisingly popular: $(0.274 - 0.248 = 0.028 > 0.02 = 0.431 - 0.429)$.*

*Note that in our example, the "most surprisingly popular" mechanism fails to identify the true state when the state is $\omega_1$, regardless of the signal of who is polled for their expectation about the population average beliefs. Indeed, if an agent who saw signal $s_2$ is polled, the true state is not even in the set of surprisingly popular states.*

*Another closely related approach proposed is that of "prediction normalized votes:" each agent votes for the state they believe is more likely (which, in this example, was the signal they saw, i.e. an agent votes for state $\omega_i$ if they see signal $s_i$). The fraction of votes each state $\omega_j$ receives is normalized by the sum, over all states $\omega_k$, the ratio of predicted vote fraction for $\omega_k$ by a $\omega_j$ voter to the predicted vote fraction for $\omega_j$ by an $\omega_k$ voter. A simple calculation shows that given the specific numerical assumption we have constructed, $\omega_2$ will have a higher prediction normalized vote than $\omega_1$ when the true state is $\omega_1$.*

### 5.2. *Other Approaches and Related Literature*

The literatures studying aggregation of information in a population is perhaps far too vast to comprehensively cite. For instance, vast literatures attempt to understand, in a variety of settings, when and how existing institutions aggregate dispersed information in strategic settings. These include studying information aggregation in voting, a vast literature with its roots in Condorcet, but more recently studied formally starting from Feddersen and Pesendorfer (1997); aggregation in auctions (see e.g. Milgrom and Weber (1982) and subsequent literature) and of course information aggregation in markets (see e.g. Grossman and Stiglitz (1980)).

We limit attention here explicitly to work that studies the design/ possibility of aggregation mechanisms/ algorithms that ignore incentives, or institutions that solely exist to aggregate information.

A key idea in the former space (indeed, in a sense an idea that is also at the heart of Prelec, Seung, and McCoy (2017)) is the Bayesian Truth Serum (Prelec (2004)— this is a procedure to truthfully elicit subjective information from agents by rewarding them as a function of others' reports. This paper pioneered the idea that agents' higher-order beliefs could be used to incentivize them successfully even if the planner/ designer did not know the prior among agents. A literature studying this followed. Prelec and Seung (2006) showed how to modify agents' reports to better aggregate them (akin to the prediction normalized voting rule we described in Example 1). Several extensions to the BTS have been proposed— see e.g. Cvitanić, Prelec, Riley, and Tereick (2019), Witkowski and Parkes (2012), Radanovic and Faltings (2013) and Radanovic and Faltings (2014). The latter three papers attempt to make the basic Bayesian truth serum robust, i.e. to ensure that it has good properties even in finite populations, and to ensure that participation is individually rational etc. Of these These ideas have also been successfully employed in real-world applications: see e.g. Shaw, Horton, and Chen (2011) for an application on Mechanical Turk, or Rigol, Hussam, and Roth (2021) who use robust BTS to identify high-ability micro-entrepreneurs by surveying peers.

On the latter, an important literature has attempted to understand the performance of prediction markets: see e.g. Wolfers and Zitzewitz (2004) for an early piece summarizing the issues and Baillon (2017) for an alternate design. Wolfers and Zitzewitz (2006) study when the prediction market price can be interpreted as the average traders' beliefs, while Ottaviani and Sørensen (2015) study when the market price underreacts to new information. Dai, Jia, and Kou (2020) show how to infer the state of the world from prediction market trading data using these theoretical ideas. There is also a large literature on pari-mutuel markets, and adapting them to aggregate information—see e.g. Pennock (2004).

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## APPENDIX A. PROOFS FROM SECTION 2.2

First we prove a general property:

**LEMMA 1.** *Suppose Part (2) of Assumption 1 is satisfied. Then for any state $\omega$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{\infty} \tilde{\mu}_i \to_P \overline{\mu}(\omega).$$

**PROOF.** The result is established in two steps. For any state $\omega$ below, we show that agents' beliefs that the state is $\omega' \in \Omega$ satisfy vanishing correlation. As beliefs lie in $[0,1]$ and therefore have bounded variance, we can apply the Bernstein law of large numbers. Together, these imply that when the state of the world is $\omega$; for any state $\omega' \in \Omega$, we have that:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{\infty} \tilde{\mu}_{i,\omega'} \to_P \overline{\mu}_{\omega'}(\omega).$$

Since there are only a finite set of states, the main result follows.

First, let us prove our claim of vanishing correlations:

**CLAIM 1.** *Conditioned on state $\omega \in \Omega$, the sequence of (random) beliefs $\langle \tilde{\mu}_{i,\omega'} \rangle_{i=1}^{\infty}$ that the state of the world is any $\omega' \in \Omega$ satisfies vanishing correlation, i.e.*

$$\lim_{|i-j|\to\infty} cov(\tilde{\mu}_{i,\omega'}, \tilde{\mu}_{j,\omega'}|\omega) = 0.$$

**PROOF OF CLAIM.** Consider agents $i$ and $j$ such that $\varepsilon$-independence among signals holds, i.e., for all events $E, E' \in \mathcal{F}$ we have

$$|P\left[\tilde{s}_i \in E, \tilde{s}_j \in E'|\omega\right] - P^S_{i,\omega}(E)P^S_{j,\omega}(E')| \leq \varepsilon.$$

Let $f_i$ denote the conditional probability function that assigns Bayesian beliefs of state $\omega'$ to signals for agent $i$. By measurability of $f_i$'s we have that every event $S, S' \subseteq [0,1]$ has a corresponding event $f_i^{-1}(S)$ and $f_j^{-1}(S')$ such that

$$\begin{aligned} &\left|\Pr\left[\tilde{\mu}_{i,\omega'} \in S, \tilde{\mu}_{j,\omega'} \in S' \,|\omega\right] - \Pr\left[\tilde{\mu}_{i,\omega'} \in S \,|\omega\right] \Pr\left[\tilde{\mu}_{j,\omega'} \in S' \,|\omega\right]\right| \\ =&\left|P\left[\tilde{s}_i \in f_i^{-1}(S), \tilde{s}_j \in f_j^{-1}(S') \,|\omega\right] - P^S_{i,\omega}(f_i^{-1}(S))P^S_{j,\omega}(f_j^{-1}(S'))\right| \end{aligned}$$

establishing approximate independence of beliefs, conditional on the state.

To establish state dependent vanishing correlation among beliefs, consider the random beliefs $\tilde{\mu}_{i,\omega'}, \tilde{\mu}_{j,\omega'}$. For any number $n > 1$, let $G_n$ denote a partition of $[0,1]$ into $n+1$ equal intervals, i.e. $G_n^k = [\frac{k-1}{n}, \frac{k}{n})$ for $k = 1$ thru $n-1$; $G_n^n = [\frac{n-1}{n}, 1]$.

Note that:

$$\mathbb{E}[\tilde{\mu}_{i,\omega'}|\omega] \leq \sum_{k=1}^{n} \frac{k}{n} \Pr(\tilde{\mu}_{i,\omega'} \in G_n^k|\omega) \equiv U_i^n,$$

$$\mathbb{E}[\tilde{\mu}_{j,\omega'}|\omega] \leq \sum_{k=1}^{n} \frac{k}{n} \Pr(\tilde{\mu}_{j,\omega'} \in G_n^k|\omega) \equiv U_j^n,$$

$$\mathbb{E}[\tilde{\mu}_{i,\omega'}\tilde{\mu}_{j,\omega'}|\omega] \leq \sum_{l=1}^{n}\sum_{k=1}^{n} \frac{lk}{n} \Pr(\tilde{\mu}_{i,\omega'} \in G_n^l, \tilde{\mu}_{j,\omega'} \in G_n^k|\omega) \equiv U_{ij}^n.$$

Since $\tilde{\mu}_{i,\omega'}$ and $\tilde{\mu}_{j,\tilde{\mu}'}$ satisfy $\varepsilon$-independence, we have that

$$|U_{ij}^n - U_i^n U_j^n| \leq \varepsilon.$$

However, as $n$ goes to infinity, by observation, $U_i^n \to \mathbb{E}[\tilde{\mu}_{i,\omega'}|\omega]$, $U_j^n \to \mathbb{E}[\tilde{\mu}_{j,\omega'}|\omega]$ and $U_{ij}^n \to \mathbb{E}[\tilde{\mu}_{i,\omega'}\tilde{\mu}_{j,\omega'}|\omega]$. Therefore $\text{cov}(\tilde{\mu}_{i,\omega'}, \tilde{\mu}_{j,\omega'}|\omega) \leq \varepsilon$.

Now, by part (2) of Assumption 1, we know that for any $i$, and $\varepsilon > 0$, $\exists n(\varepsilon)$ large enough so that if $j - i > n$ so that we have that conditional on state, $\tilde{\mu}_i$ and $\tilde{\mu}_j$ are $\varepsilon$-independent. The result trivially follows. ■

Now, recall the Bernstein law of large numbers (See e.g 3.3.2 of Shiryaev (2012)):

**THEOREM 5.** *Let $\tilde{X}_i$, $i = 1,2,3\ldots$ be a sequence of real valued random variables such that (1) $\mathbb{E}[\tilde{X}_i]$ is finite for each $i$, (2) there exists $K$ finite such that $\mathbb{V}[\tilde{X}_i] \leq K$ for all $i$, and (3) $\lim_{|i-j|\to\infty} cov(X_i, X_j) = 0$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(\tilde{X}_i - \mathbb{E}[X_i]\right) \to_P 0.$$

Observe that since $\tilde{\mu}_{i,\omega'}$ is a belief, it lies between $[0,1]$ and so the first two conditions of the Theorem are verified. Claim 1 verifies the third condition conditional on any true state $\omega$. Therefore we have the desired result for any $\omega' \in \Omega$, conditional on true state $\omega$. Since the set of states is finite, we have convergence in probability of the entire vector. ■

**LEMMA 2.** *Suppose $L = 2$. Suppose further that part (1) of Assumption 1 holds and $P_{i,\omega_1}^S \neq P_{i,\omega_2}^S$. Then following statements hold:*

*(1) $\frac{\hat{G}_{i,\omega_1}(r)}{\hat{G}_{i,\omega_2}(r)}$ is non-increasing in $r$*

*(2) $\frac{\hat{G}_{i,\omega_1}(r)}{\hat{G}_{i,\omega_2}(r)} > 1$ for all $r \in \left(\beta, \bar{\beta}\right)$ where $\left[\underline{\beta}, \bar{\beta}\right]$ denote the convex hull of the support of private beliefs.*[4]

*(3) $\overline{\mu}_i(\omega_1) < \overline{\mu}_i(\omega_2)$.*

**PROOF.** By Bayes' rule, we know that

$$r = \frac{dG_{i,\omega_2}(r)\, P(\omega_2)}{dG_{i,\omega_1}(r)\, P(\omega_1) + dG_{i,\omega_2}(r)\, P(\omega_2)}. \tag{5}$$

[4]Since signal generating measures are absolutely continuous each other, the support is identical across states.

We define $\alpha = P(\omega_2)/P(\omega_1)$ as the relative likelihood of the states according to the prior $P$. Since no signal is completely informative, i.e., $r \notin \{0,1\}$, we can rewrite (5) as

$$\frac{dG_{i,\omega_1}}{dG_{i,\omega_2}}(r) = \left(\frac{1-r}{r}\right)\alpha. \tag{6}$$

Hence, for any $r \in (0,1)$, we have

$$G_{i,\omega_1}(r) = \int_0^r dG_{i,\omega_1}(x) = \int_0^r \left(\frac{1-x}{x}\right)\alpha dG_{i,\omega_2}(x) \geq \left(\frac{1-r}{r}\right)\alpha G_{i,\omega_2}(r). \tag{7}$$

*Part (1)*. Pick any $r', r$ s.t. $r' > r$.Observe that:

$$\begin{aligned} & \frac{G_{i,\omega_1}(r)}{G_{i,\omega_2}(r)} - \frac{G_{i,\omega_1}(r')}{G_{i,\omega_2}(r')}, \\ = & \frac{G_{i,\omega_1}(r)G_{i,\omega_2}(r') - G_{i,\omega_1}(r')G_{i,\omega_2}(r)}{G_{i,\omega_2}(r)G_{i,\omega_2}(r')}. \end{aligned}$$

Define $\Delta_{i,\omega} = G_{i,\omega}(r') - G_{i,\omega}(r)$. We then have:

$$= \frac{G_{i,\omega_1}(r)\Delta_{i,\omega_2} - \Delta_{i,\omega_1}G_{i,\omega_2}(r)}{G_{i,\omega_2}(r)G_{i,\omega_2}(r')},$$

Observe that $\Delta_{i,\omega_1} = \int_r^{r'} dG_{i,\omega_1}(x) = \int_r^{r'} \left(\frac{1-x}{x}\right)\alpha dG_{i,\omega_2}(x) \leq \left(\frac{1-r}{r}\right)\alpha\Delta_{i,\omega_2}$. Substituting in,

$$\begin{aligned} \geq & \frac{G_{i,\omega_1}(r)\Delta_{i,\omega_2} - \frac{1-r}{r}\alpha\Delta_{i,\omega_2}G_{i,\omega_2}(r)}{G_{i,\omega_2}(r)G_{i,\omega_2}(r')}, \\ \geq & 0, \end{aligned}$$

where the last inequality follows by substituting in $G_{i,\omega_1}(r)$ in the numerator from (7).

*Part (2)*. It follows from (7) that

$$G_{i,\omega_1}(r) \geq G_{i,\omega_2}(r) \text{ if } \left(\frac{1-r}{r}\right)\alpha \geq 1. \tag{8}$$

If $\left(\frac{1-r}{r}\right)\alpha < 1$, then

$$\begin{aligned} 1 - G_{i,\omega_1}(r) &= \int_r^1 dG_{i,\omega_1}(x) \\ &= \int_r^1 \left(\frac{1-x}{x}\right)\alpha dG_{i,\omega_2}(x) \\ &\leq \left(\frac{1-r}{r}\right)\alpha\left(1 - G_{i,\omega_2}(r)\right) \\ &\leq 1 - G_{i,\omega_2}(r). \end{aligned} \tag{9}$$

Combining (8) and (9), we obtain that

$$\frac{G_{i,\omega_1}(r)}{G_{i,\omega_2}(r)} \geq 1. \tag{10}$$

To prove $\frac{G_{i,\omega_1}(r)}{G_{i,\omega_2}(r)} > 1$ for any $r \in \left(\underline{\beta}, \overline{\beta}\right)$, suppose to the contrary that $\frac{G_{i,\omega_1}(r)}{G_{i,\omega_2}(r)} = 1$ for some $r \in \left(\underline{\beta}, \overline{\beta}\right)$. We derive contradiction in each of the following two cases

**Case 1:** $\left(\frac{1-r}{r}\right)\alpha < 1$. Then,

$$\begin{aligned} G_{i,\omega_1}(1) &= G_{i,\omega_1}(r) + \int_r^1 dG_{i,\omega_1}(x) \\ &= G_{i,\omega_2}(r) + \int_r^1 dG_{i,\omega_1}(x) \\ &= G_{i,\omega_2}(r) + \int_r^1 \left(\frac{1-x}{x}\right)\alpha dG_{i,\omega_2}(x) \\ &\leq G_{i,\omega_2}(r) + \left(\frac{1-r}{r}\right)\alpha \int_r^1 dG_{i,\omega_2}(x) \\ &= G_{i,\omega_2}(r) + \left(\frac{1-r}{r}\right)\alpha \left[1 - G_{i,\omega_2}(r)\right] \end{aligned}$$

which yields a contradiction unless $G_{i,\omega_2}(r) = 1$. However, $G_{i,\omega_2}(r) = 1$ implies $r \geq \overline{\beta}$ which contradicts $r \in \left(\underline{\beta}, \overline{\beta}\right)$.

**Case 2:** $\left(\frac{1-r}{r}\right)\alpha \geq 1$ . Then, since $\frac{G_{i,\omega_1}(r)}{G_{i,\omega_2}(r)} = 1$ and $\frac{G_{i,\omega_1}(r)}{G_{i,\omega_2}(r)}$ is non-increasing in $r$, together with (10) implies that $\frac{G_{i,\omega_1}(r')}{G_{i,\omega_2}(r')} = 1$ for all $r' \geq r$. Pick $r' > \frac{\alpha}{1+\alpha}$ (note that since $\frac{\alpha}{1+\alpha}$ is just the prior belief that $\omega_1 = 1$, it must be in the interior of the support of beliefs $[\underline{\beta}, \bar{\beta}]$—this leads to the contradiction in Case 1.

*Part (3)*. This follows directly from the first-order stochastic dominance relationship between $G_{i,\omega_1}$ and $G_{i,\omega_2}$ in established part (2). ■

**LEMMA 3.** *Suppose $L = 2$ and the common prior $P$ satisfies both Assumptions 1 and 2. Then, there exists an $\varepsilon > 0$ such that the $\overline{\mu}_i(\omega_1) + \varepsilon < \overline{\mu}_i(\omega_2)$ for all $i \in N$.*

**PROOF.** Fix any agent $i$ and let the convex hull of the common support of its posteriors be $[\underline{\beta}, \overline{\beta}]$. First observe that the prior belief about the state is $\frac{\alpha}{1+\alpha}$. Therefore it must be that $\underline{\beta} < \frac{\alpha}{1+\alpha} < \overline{\beta}$ (if exactly one of these is an equality we violate the condition that the expected posterior is the prior, if both are equalities then the signal is uninformative violating Assumption 2.

**CLAIM 2.** *For $\delta > 0$ if the total variation distance between $G_{i,\omega_1}$ and $G_{i,\omega_2}$ is larger than $\delta$, there exists $\delta' > 0$ such that $\underline{\beta} + \delta' < \frac{\alpha}{1+\alpha} < \overline{\beta} - \delta'$, and further* $\max\left\{G_{i,\omega_1}(\frac{\alpha}{1+\alpha} - \delta'), (1 - G_{i,\omega_2}(\frac{\alpha}{1+\alpha} + \delta')\right\} \geq \delta'$.

**PROOF.** To see this observe that we showed in the proof of Lemma 2 that for any $r \notin \{0,1\}$, we have

$$\frac{dG_{i,\omega_1}}{dG_{i,\omega_2}}(r) = \left(\frac{1-r}{r}\right)\alpha.$$

Therefore for $r$ arbitrarily close to $\frac{\alpha}{1+\alpha}$, $\frac{dG_{i,\omega_1}}{dG_{i,\omega_2}}(r)$ is arbitrarily close to 1. Further, by the fact that $G_{i,\omega_2}$ first order stochastically dominates $G_{i,\omega_1}$, we have $\delta' > G_{i,\omega_1}(\frac{\alpha}{1+\alpha} - \delta') > G_{i,\omega_2}(\frac{\alpha}{1+\alpha} - \delta')$, and similarly $\delta' > (1 - G_{i,\omega_2}(\frac{\alpha}{1+\alpha} + \delta') > (1 - G_{i,\omega_1}(\frac{\alpha}{1+\alpha} + \delta')$

Therefore if the claim is not satisfied, both measures are mostly supported on $[\frac{\alpha}{1+\alpha} - \delta', \frac{\alpha}{1+\alpha} + \delta']$ the total variation distance is $\leq 2\frac{1+\alpha}{\alpha}\delta'^2 + 2\delta'$. Therefore the claim follows for e.g. $\delta' = \frac{1}{3}\delta$. ■

Now observe that $\overline{\mu}_i(\omega) = \int_0^1 (1 - G_{i,\omega}(r))dr$. Therefore

$$\begin{aligned}&\overline{\mu}_i(\omega_2) - \overline{\mu}(\omega_1) \\ =&\int_0^1 (G_{i,\omega_1}(r) - G_{i,\omega_2}(r))dr\end{aligned}$$

We showed in Lemma 2 that $G_{i,\omega_2}$ first order stochastically dominates $G_{i,\omega_1}$. Therefore we have, for $\delta'$ that satisfies the statement of the claim above,

$$\begin{aligned}\geq& \int_{\frac{\alpha}{1+\alpha}-\delta'}^{\frac{\alpha}{1+\alpha}} (G_{i,\omega_1}(r) - G_{i,\omega_2}(r))dr \\ =&\delta'(G_{i,\omega_1}(\frac{\alpha}{1+\alpha} - \delta') - G_{i,\omega_2}(\frac{\alpha}{1+\alpha} - \delta')) \\ &+ \int_{\frac{\alpha}{1+\alpha}-\delta'}^{\frac{\alpha}{1+\alpha}} \left(\int_{\frac{\alpha}{1+\alpha}-\delta'}^{r} (dG_{i,\omega_1}(x) - dG_{i,\omega_2}(x))\right) dr\end{aligned}$$

We know from (6) that $dG_{i,\omega_1}(r) > dG_{i,\omega_2}(r)$ for $r < \frac{\alpha}{1+\alpha}$. So we have,

$$\geq \delta'(G_{i,\omega_1}(\frac{\alpha}{1+\alpha} - \delta') - G_{i,\omega_2}(\frac{\alpha}{1+\alpha} - \delta')).$$

Now observe that by (7) we have

$$\begin{aligned}&G_{i,\omega_1}(r) \geq \frac{1-r}{r}\alpha G_{i,\omega_2}(r), \\ \Longrightarrow\ &G_{i,\omega_1}(r) - G_{i,\omega_2}(r) \geq \left(\frac{1-r}{r}\alpha - 1\right) G_{i,\omega_2}(r).\end{aligned}$$

For $r = \frac{\alpha}{1+\alpha} - \delta'$ we have

$$\implies G_{i,\omega_1}(\frac{\alpha}{1+\alpha} - \delta') - G_{i,\omega_2}(\frac{\alpha}{1+\alpha} - \delta') \geq \frac{\delta'(\alpha+1)^2}{\alpha - (\alpha+1)\delta'} G_{i,\omega_2}(\frac{\alpha}{1+\alpha} - \delta').$$

However we also have by the statement of the claim that

$$G_{i,\omega_1}(\frac{\alpha}{1+\alpha} - \delta') \geq \delta',$$
$$\implies G_{i,\omega_1}(\frac{\alpha}{1+\alpha} - \delta') - G_{i,\omega_2}(\frac{\alpha}{1+\alpha} - \delta') \geq \delta' - G_{i,\omega_2}(\frac{\alpha}{1+\alpha} - \delta').$$

Combining, we have that

$$\geq \max\{\delta' - G_{i,\omega_2}(\frac{\alpha}{1+\alpha} - \delta'), \frac{\delta'(\alpha+1)^2}{\alpha - (\alpha+1)\delta'} G_{i,\omega_2}(\frac{\alpha}{1+\alpha} - \delta')\}$$

Note that the first term is decreasing in $G_{i,\omega_2}(\frac{\alpha}{1+\alpha} - \delta')$, while the second is increasing. Therefore, the minimum is achieved at

$$\delta' - G_{i,\omega_2}(\frac{\alpha}{1+\alpha} - \delta') = \frac{\delta'(\alpha+1)^2}{\alpha - (\alpha+1)\delta'} G_{i,\omega_2}(\frac{\alpha}{1+\alpha} - \delta')$$
$$\implies G_{i,\omega_2}(\frac{\alpha}{1+\alpha} - \delta') = \frac{\alpha - (\alpha+1)\delta'}{\alpha + \alpha\delta'(\alpha+1)} \delta'$$

Substituting in we have that,

$$G_{i,\omega_1}(\frac{\alpha}{1+\alpha} - \delta') - G_{i,\omega_2}(\frac{\alpha}{1+\alpha} - \delta') \geq \frac{\delta'(\alpha+1)^2}{\alpha + \alpha\delta'(\alpha+1)} \delta',$$

and therefore we have that $\overline{\mu}_i(\omega_2) - \overline{\mu}_i(\omega_1) \geq \varepsilon$ for $\varepsilon = \frac{\delta'(\alpha+1)^2}{\alpha+\alpha\delta'(\alpha+1)} \delta'^2$. ■

We are now finally in a position to give the proof of Theorem 1.

**Theorem 1.** *Suppose the information structure among agents satisfies Assumptions 1 and 2. Then* population mean based aggregation *procedure (Procedure 1) correctly recovers the true state of the world a.s.*

**Proof of Theorem 1.** We prove this by analyzing each of the steps of Procedure 1.

(1) By Lemma 1, a law of large numbers holds and when the realizes state is $\omega$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \tilde{\mu}_i = \overline{\mu}(\omega).$$

Therefore the population mean of the elicited beliefs $\hat{\mu}$ in Step (1) of the procedure equals $\overline{\mu}(\omega)$ in state $\omega$.

(2) By part (1) of Assumption 1, we know that the beliefs of any agent $i$ have a common support across the states. Further, by Assumption 2, this support must have

at least 2 distinct points in it. Combining this with the fact that signals are approximately independent, there must exist at least two agents with different beliefs almost surely. So step 2 can find 2 such agents. Further, in step 2, each agent $i = A, B$ will report, by the law of total probability: $\alpha_i = \mu_i \overline{\mu}(\omega_1) + (1 - \mu_i)\overline{\mu}(\omega_2)$.

(3) Since $\mu_A \neq \mu_B$, equation 1 can be solved.

(4) By Lemma 3, there is a $\varepsilon > 0$ such that for all $i$ $\overline{\mu}_i(\omega_1) < \overline{\mu}_i(\omega_2) - \varepsilon$. Therefore, the expected poptulation average beliefs, $\overline{\mu}(\omega_1) < \overline{\mu}(\omega_2) - \varepsilon$, i.e. $\overline{\mu}(\omega_1) \neq \overline{\mu}(\omega_2)$.
Therefore comparing the elicited $\hat{\mu}$ in step (1) with the recovered $(\overline{\mu}(\omega_1), \overline{\mu}(\omega_2))$ must reveal the true state. ∎

**THEOREM 2.** *Suppose $L > 2$ and the common prior $P$ satisfies Assumptions 1, 3 and 4. Then the PMBA procedure (Procedure 2) recovers the true unknown state of nature $\omega$ almost surely.*

**PROOF.** We prove this by analyzing each of the steps of Procedure 2 analogously to our proof of Theorem 1.

(1) By Lemma 1, a law of large numbers holds and when the realizes state is $\omega$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \tilde{\mu}_i = \overline{\mu}(\omega).$$

Therefore the population mean of the elicited beliefs $\hat{\mu}$ in Step (1) of the procedure equals $\overline{\mu}(\omega)$ in state $\omega$.

(2) By part (1) of Assumption 1, we know that the beliefs of any agent $i$ have a common support across the states. Further, by Assumption 4, this support must has an interior relative to $\Delta^L$. Combining this with the fact that signals are approximately independent, there must exist at least $L$ agents whose beliefs form a full rank matrix a.s.. So step 2 can find $L$ such agents. Further, in step 2, each agent $i \in I$ will report, by the law of total probability: $\alpha_i = \mu_i^T \bar{\boldsymbol{\mu}}$.

(3) Since the set $I$ of agents is selected so that $\boldsymbol{\mu}$ is full-rank, equation 2 can be solved.

(4) By Assumption 3, $\overline{\mu}(\omega) \neq \overline{\mu}(\omega')$ for any $\omega, \omega' \in \Omega$.
Therefore comparing the elicited $\hat{\mu}$ in step (1) with the recovered $\boldsymbol{m\bar{u}}$ must reveal the true state. ∎

## APPENDIX B. PROOFS FROM SECTION 3

We begin with a helpful lemma. First, let us first introduce some notation. Recall that $N = \{1, 2, 3, ...\}$ is the infinite set of agents. Let $N_n$ be the set of the first $n$ agents in $N$,

$$N_n = \{1, ..., n\}.$$

**LEMMA 4.** *Let $\langle \zeta_i \rangle_{i\in N}$ be a sequence of independent random variables $E[\zeta_i] = 0$ and let $\langle x_i \rangle_{i\in N}$ be a sequence of i.i.d. indicator random variables distributed according to $x$ with $E[x] > 0$ such* *that $\zeta_i$ and $x_i$ are independent for all $i \in N$. Let $N_n^1 \subseteq N_n$ be the random subset of agents with $x_i = 1$. Consider the random variable*

$$\rho_n = \begin{cases} \frac{1}{|N_n^1|}\sum_{i \in N_n^1} \zeta_i & \text{if } |N_n^1| > 0, \\ c & \text{otherwise.} \end{cases}$$

*Where $c \in \mathbb{R}$. Then $\rho_n \to 0$ almost surely.*

**PROOF.** Note that since $E[x] > 0$, $|N_n^1| \to \infty$ almost surely as $n \to \infty$. Fix any realization of $x_i$'s such that $N_n^1 \to \infty$. Consider the subsequence of $i's$ such that $x_i = 1$. Since $\zeta_i$'s are independent of $x_i$'s by assumption, the limit average of $\zeta_i$'s on this subsequence must converge to the limit average of $E[\zeta_i]$ by the strong law of large numbers. Since $E[\zeta_i] = 0$ by assumption, we have that

$$\lim_{n \to \infty} \frac{1}{|N_n^1|} \sum_{i \in N_n^1} \zeta_i = 0,$$

for this realization of $x_i$'s. Since such realizations (i.e. ones where $|N_n^1| \to \infty$) occur almost surely, we have that $\rho_n \to 0$ a.s. as desired. ■

We are now in a position to provide a proof of the main theorem, Theorem 3 (restated below for the reader's convenience).

**THEOREM 3.** *Suppose the true signal distribution process satisfies Assumptions 1 and 2, and that the agents' knowledge regarding the information structure satisfies Assumption 5. Then* Limited Information Population Mean Based Aggregation *(Procedure 3) correctly recovers the true state of the world almost surely.*

**PROOF.** Consider a realization of the first order reports. Let groups $A$ and $B$ be as defined in Procedure 3. By our maintained assumptions, each of these groups must have an infinite number of members almost surely. Part (1) of Assumption 1 implies that the agents' beliefs must lie in $(0, 1)$. Therefore the distribution of beliefs conditional on state must have at least two points in their support. Part (2) implies that (most) agents' beliefs are approximately independent conditional on state. The result follows.

For agent $i \in A$, let $\alpha_i$ denote her expectation of the population mean. Given her first order belief $\mu_i$ and pair of state-dependent population beliefs $(\alpha_i^0, \alpha_i^1)$ she reports

$$\alpha_i = \mu_i \alpha_i^1 + (1 - \mu_i)\alpha_i^2.$$

By Assumption 5 this can be rewritten as

$$\alpha_i = \mu_i \overline{\mu}(\omega_1) + (1 - \mu_i)\overline{\mu}(\omega_2) + \left(\mu_i \zeta_i^{\omega_1} + (1 - \mu_i)\zeta_i^{\omega_2}\right).$$

Denote by $A^l$ the set of first $l$ agents in $A$ according to the order on $N$. Consider the average over the reports of the agents in $A^l$

$$\frac{1}{l}\sum_{i\in A^l}\alpha_i = \frac{1}{l}\sum_{i\in A^l}\left((1-\mu_i)\alpha_i^1 + \mu_i\alpha_i^2\right),$$
$$= \frac{1}{l}\sum_{i\in A^l}\left(\mu_i\overline{\mu}(\omega_1) + (1-\mu_i)\overline{\mu}(\omega_2) + \left(\mu_i\zeta_i^{\omega_1} + (1-\mu_i)\zeta_i^{\omega_2}\right)\right)$$

By the conclusion of Lemma 4, we therefore have that

$$\lim_{l\to\infty}\frac{1}{l}\sum_{i\in A^l}\alpha_i = \lim_{l\to\infty}\frac{1}{l}\sum_{i\in A^l}\left(\mu_i\overline{\mu}(\omega_1) + (1-\mu_i)\overline{\mu}(\omega_2)\right)$$

Therefore, we have that

$$\alpha_A \equiv \lim_{l\to\infty}\frac{1}{l}\sum_{i\in A^l}\alpha_i = \mu_A\overline{\mu}(\omega_1) + (1-\mu_A)\overline{\mu}(\omega_2).$$

Similarly, for the agents in $B$ we have that

$$\alpha_B \equiv \lim_{l\to\infty}\frac{1}{l}\sum_{i\in B^l}\alpha_i = \mu_B\overline{\mu}(\omega_1) + (1-\mu_B)\overline{\mu}(\omega_2),$$

where $\mu_A$ and $\mu_B$ are as defined in Procedure 3. As before, we can recover $\overline{\mu}(\omega_1), \overline{\mu}(\omega_2)$ as in (3):

$$\begin{pmatrix}\overline{\mu}(\omega_1)\\ \overline{\mu}(\omega_2)\end{pmatrix} = \begin{pmatrix}\mu_A & 1-\mu_A\\ \mu_B & 1-\mu_B\end{pmatrix}^{-1}\begin{pmatrix}\alpha_A\\ \alpha_B\end{pmatrix}.$$

Note that $\alpha_A, \alpha_B$ are directly elicited, while $\mu_A \neq \mu_B$ by construction implies the matrix is invertible. By Lemma 1) we have that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mu_i = \overline{\mu}(\omega)$$

almost surely, and further that $\overline{\mu}(\omega_1) \neq \overline{\mu}(\omega_2)$. Therefore by comparing the realized average of elicited beliefs, to the $\overline{\mu}(\cdot)$ recovered from (3), we can almost surely recover the true state of the world, $\omega$, as desired. ■

## Appendix C. Proofs from Section 4

**Theorem 4.** *Suppose that $\tilde{\mu}$ has a finite support and the agents report $\tilde{s} = (\tilde{s}_i)_i$. Then there exists a procedure which recovers the "pooled information" posterior on states, i.e. $\tilde{\mu}(\cdot|\tilde{s})$.*

**Proof.** Assume that $\text{marg}_{\tilde{S}}\tilde{\mu}\,(\tilde{s}) > 0$. It follows from Theorem III.2.7 of Mertens, Sorin, and Zamir (2015) that the aggregator can derive:

(1) The set $E(\tilde{s})$ which is the smallest set $Y \subseteq \Omega \times \tilde{S}$ of state-hierarchy profiles satisfying
   (a) $(\omega, \tilde{s}) \in Y$ for some $\omega \in \Omega$, and,
   (b) for each $(\omega, \tilde{t}) \in Y$, we have $\{\tilde{t}_i\} \times \text{supp } \tilde{\pi}_i(\tilde{t}_i) \subseteq Y$.

(2) The unique consistent probability $\tilde{\mu}$ on $E(\tilde{s})$.

With (1) and (2), the aggregator can compute $\tilde{\mu}(\cdot|\tilde{s})$. We recap and illustrate both (1) and (2) for the ease of reference.

To construct (1), define $C_i^1(\omega, \tilde{t}) = \{\tilde{t}_i\} \times \text{supp}\tilde{\pi}_i(\tilde{t}_i)$ for each $(\omega, \tilde{t}) \in \Omega \times \tilde{S}$; and inductively, for every $l \geq 1$ and $\tilde{t} \in \tilde{S}$, define

$$C_i^{l+1}(\omega, \tilde{t}) = C_i^l(\omega, \tilde{t}) \bigcup \bigcup_{(\omega', \tilde{t}') \in C_i^l(\omega, \tilde{t})} \bigcup_{j=1}^{N} C_j^1(\omega', \tilde{t}').$$

Then, let

$$C_i(\omega, \tilde{t}) = \bigcup_{l=1}^{\infty} C_i^l(\omega, \tilde{t}).$$

These include $(\omega, \tilde{t})$, the state-hierarchy profiles which agent $i$ regards as possible (i.e., $C_i^1(\omega, \tilde{t})$) at $(\omega, \tilde{t})$, the state-hierarchy profiles which some agent regards at some state-hierarchy profile in $C_i^1(\omega, \tilde{t})$ (i.e., $C_i^2(\omega, \tilde{t})$), and so on.

Consider any $\omega^*$ such that $\tilde{\mu}(\omega^*, \tilde{s}) > 0$. Note that $C_i(\omega^*, \tilde{s})$ satisfies properties (a) and (b) above by construction and hence $C_i(\omega^*, \tilde{s}) \supseteq E(\tilde{s})$.[5] Also we can argue that $C_i(\omega^*, \tilde{s}) \subseteq E(\tilde{s})$ inductively.

For (2), it follows from Bayes' rule that for every $(\omega', \tilde{t}') \in C_i^1(\omega, \tilde{t})$ and $\tilde{\mu}(\omega, \tilde{t}) > 0$, we must have $\tilde{t}_i = \tilde{t}'_i$ and hence

$$\frac{\tilde{\mu}(\omega', \tilde{t}')}{\tilde{\mu}(\omega, \tilde{t})} = \frac{\tilde{\pi}_i(\tilde{t}'_i)(\omega', \tilde{t}')\,\tilde{\mu}_i(\tilde{t}'_i)}{\tilde{\pi}_i(\tilde{t}_i)(\omega, \tilde{t}_{-i})\,\tilde{\mu}_i(\tilde{t}_i)} = \frac{\tilde{\pi}_i(\tilde{t}'_i)(\omega', \tilde{t}')}{\tilde{\pi}_i(\tilde{t}_i)(\omega, \tilde{t}_{-i})} > 0.$$

The aggregator knows $\tilde{\pi}_i(\tilde{t}_i)$ and hence can express $\tilde{\mu}(\omega', \tilde{t}')$ as a multiple of $\tilde{\mu}(\omega, \tilde{t})$. Inductively, for every $(\omega', \tilde{t}') \in C_i^l(\omega, \tilde{t})$, $\tilde{\mu}(\omega', \tilde{t}')$ can also be expressed as a multiple of $\tilde{\mu}(\omega, \tilde{t})$.

Hence, $\tilde{\mu}$ is uniquely determined on $E(\tilde{s}) = C_i(\omega^*, \tilde{s})$ since $\tilde{\mu}(\omega^*, \tilde{s}) > 0$. $\tilde{\mu}(\cdot|\tilde{s})$ can thus be calculated. ■

## C.1. *Construction for $m > 2$*

Similarly, we can generalize the construction in Section 4.2 to the case with $m \geq 3$ as follows:

$$\Pi_1 = \left\{ \left\{ \{(\sigma_1, 2k-1), (\sigma_1, 2k), (\sigma_2, k)\} : k = 1, ..., 2^{m-1} \right\}, \left\{ \left(\sigma_2, 2^{m-1}+1\right), ...., (\sigma_2, 2^m) \right\} \right\};$$

[5] Property (b) is by construction and property (a) follows from the assumption that $\tilde{\mu}(\omega^*, \tilde{s}) > 0$.

$$\Pi_2 = \left\{ \left\{ \{(\sigma_2, 2k-1), (\sigma_2, 2k), (\sigma_1, k)\} : k = 1, ..., 2^{m-1} \right\}, \left\{ \left(\sigma_1, 2^{m-1}+1\right), ...., (\sigma_1, 2^m) \right\} \right\}.$$

$$\mu(\sigma_l, k) = \frac{1}{2^{m+1}}, \forall l = 1, 2, \forall k = 1, 2, ..., 2^m.$$

Without loss of generality, assume that $m \geq 3$ is odd. Construct the new model as follows:

$$\Pi_1' = \left\{ \begin{array}{c} \{(\sigma_1, 1), (\sigma_2, 1), (\sigma_1, 2)\}, \{(\sigma_1, 1)', (\sigma_2, 2)', (\sigma_1, 3)', (\sigma_1, 4)'\}, \\ \{\{(\sigma_1, 2k-1)', (\sigma_1, 2k)', (\sigma_2, k)'\} : k = 2^{n-1}+1, ..., 2^n \text{ and } n = 3, 5, ..., m-2\}, \\ \{\{(\sigma_1, 2k-1), (\sigma_1, 2k), (\sigma_2, k)\} : k = 2^{n-1}+1, ..., 2^n \text{ and } n = 2, 4, ..., m-1\}, \\ \left\{ \left(\sigma_2, 2^{m-1}+1\right)', ...., (\sigma_2, 2^m)' \right\} \end{array} \right\};$$

$$\Pi_2' = \left\{ \begin{array}{c} \{(\sigma_1, 1), (\sigma_2, 1), (\sigma_1, 1)', (\sigma_2, 2)'\}, \\ \{\{(\sigma_2, 2k-1)', (\sigma_2, 2k)', (\sigma_1, k)'\} : k = 2^{n-1}+1, ..., 2^n \text{ and } n = 2, 4, ..., m-1\}, \\ \{\{(\sigma_2, 2k-1), (\sigma_2, 2k), (\sigma_1, k)\} : k = 2^{n-1}+1, ..., 2^n \text{ and } n = 1, 3, ..., m-2\}, \\ \left\{ \left(\sigma_1, 2^{m-1}+1\right), ...., (\sigma_1, 2^m) \right\} \end{array} \right\}.$$

As in the case of $m = 2$, we start from the partition cells $\{(\sigma_1, 1), (\sigma_2, 1), (\sigma_1, 2)\}$ and $\left\{(\sigma_1, 1), (\sigma_2, 1), (\sigma_1, 1)', (\sigma_2, 2)'\right\}$ which contain $(\sigma_1, 1)$. The states with $'$ are those "to the left" of $(\sigma_1, 1)$, whereas the states without $'$ are like those "to the right" of $(\sigma_1, 1)$. We can then mimic the idea for $m = 2$ to solve for a prior $\mu'$ with the desired properties as follows:

(1) Again, for $x > 0$, set

$$\mu'(\sigma_1, 1) = 0 \text{ and } \mu'(\sigma_2, 1) = \mu'(\sigma_1, 1)' = \mu'(\sigma_2, 2)' = x.$$

(2) The number of states with $'$ excluding $(\sigma_1, 1)'$ and $(\sigma_2, 2)'$ (i.e., states in $\Pi_2(\sigma_1, k)'$ where $k = 2^{n-1}+1, ..., 2^n$ and $n = 2, 4, ..., m-1$) is :

$$y \equiv 3 \times \left(2^1 + 2^3 + \cdots + 2^{m-2}\right),$$
$$\text{i.e., } y = 2^m - 2.$$

We assign probability $\frac{x}{2}$ to each of these "left" states.

(3) The number of states without $'$ excluding $(\sigma_1, 1)$ and $(\sigma_2, 1)$ (i.e., $(\sigma_2, 2)$ and states in $\Pi_1(\sigma_2, k)$ where $k = 2^{n-1}+1, ..., 2^n$ and $n = 2, 4, ..., m-1$) is:

$$1 + 3 \times \left(2^1 + 2^3 + \cdots + 2^{m-2}\right) = y + 1.$$

We assign probability $2x$ to each of the "right" states.

(4) Hence for the total probability to sum to 1, we must have:

$$3x + y\frac{x}{2} + (y+1)2x = 1,$$
$$\implies x = \frac{2}{10 + 5y},$$

$$\Longrightarrow x = \frac{1}{5 \times 2^m}.$$

In summary, observe that as desired, we have that

$$\begin{aligned} \mu\left(\omega_1 | \left(\Pi_1(\sigma_1,1), \Pi_2(\sigma_1,1)\right)^m\right) &= \frac{1}{2}. \\ \mu'\left(\omega_1 | \left(\Pi_1'(\sigma_1,1), \Pi_2'(\sigma_1,1)\right)^m\right) &= 0. \\ \left(\Pi_1'(\sigma_1,1), \Pi_2'(\sigma_1,1)\right)^m &= \left(\Pi_1(\sigma_1,1), \Pi_2(\sigma_1,1)\right)^m. \end{aligned}$$

Clearly, we can flip left and right to construct another model $\Pi_1''$ and $\Pi_2''$ with

$$\begin{aligned} \mu'\left(\omega_1 | \left(\Pi_1''(\sigma_1,1), \Pi_2''(\sigma_1,1)\right)^m\right) &= 1; \\ \left(\Pi_1''(\sigma_1,1), \Pi_2''(\sigma_1,1)\right)^m &= \left(\Pi_1(\sigma_1,1), \Pi_2(\sigma_1,1)\right)^m. \end{aligned}$$

The construction here makes use of the feature that all higher-order beliefs of the common knowledge hierarchy $t$ are degenerate. More precisely, this feature ensures that in "shifting" out the probability which $\mu$ assigns to $(\sigma_1, 1)$, we can preserve the higher-order beliefs of $\Pi_1(\sigma_1, 1)$ and $\Pi_2(\sigma_1, 1)$ as long as we can preserve the first-order belief at every partition cell except for $\Pi_1(\sigma_2, 2^m)$ and $\Pi_2(\sigma_1, 2^m)'$.